Classification of Minimal Singularity Thresholds

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Abstract

In 2014, Demailly and Pham [5] gave a sharp lower bound on the log canonical threshold of a finite-colength ideal $I \subset \mathbb{C}\{x_1, \ldots, x_n\}$ in terms of the mixed multiplicities of I. We give an analogous lower bound on the F-pure threshold in positive characteristics. In equal characteristic, we show that the class of homogeneous ideals realizing the minimum admits a simple classification.

Log Canonical Threshold

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $\quad I = (f_1, \ldots, f_r) \subseteq (\underline{x})$.
The *log canonical threshold* of I at 0 is a positive number which measures the singularities of (R, I):

$$\operatorname{lct}(I) = \sup \left\{ \begin{array}{c} \lambda > 0 : (|f_1|^2 + \cdots + |f_r|^2)^{-\lambda} \\ \text{is locally integrable at } 0. \end{array} \right\}$$

Properties of the LCT

1. The log canonical threshold can be computed from the data of a log resolution of (R, I).
2. $\operatorname{lct}(I)$ does not depend on the choice of generators $f_1, \ldots, f_r$.
3. $\operatorname{lct}(I) \in \mathbb{Q} \cap (0, \operatorname{codim}(I)]$
4. R/I smooth at $0 \implies \operatorname{lct}(I) = \operatorname{codim}(I)$.
5. $I \subseteq J \implies \operatorname{lct}(I) \leq \operatorname{lct}(J)$.
6. $\operatorname{lct}(I) = \operatorname{lct}(\bar{I})$, where $\bar{I}$ is the integral closure.

F-Pure Threshold

Let k be a field of characteristic $p > 0$.
Set $R = k[x_1, \ldots, x_n]$, $\quad \mathfrak{m} = (x_1, \ldots, x_n) \supseteq I$.
The *F-pure threshold* of I at $\mathfrak{m}$ is a positive number which measures the F-singularities of the pair (R, I).

$$\operatorname{fpt}(I) = \sup \left\{ \frac{a}{p^e} : I^a \not\subseteq \mathfrak{m}^{[p^e]} \right\}.$$

The fpt satisfies properties analogous to (3)-(6).

Notation and Conventions

Fix the following conventions.

- k denotes an algebraically closed field
- $R = k[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$
- $I \subseteq R$ is a homogeneous $\mathfrak{m}$-primary ideal.

Important Convention

$$c(I) = \begin{cases} \operatorname{lct}(I) & \text{char } k = 0 \\ \operatorname{fpt}(I) & \text{char } k > 0 \end{cases}$$

Mixed Multiplicities and the Demailly-Pham Invariant

There are $e_0(I), \ldots, e_n(I) \in \mathbb{Z}^+$ s.t. for $r, s \in \mathbb{Z}^+$:

$$n! \cdot \operatorname{length}\left(\frac{R}{I^r \mathfrak{m}^s}\right) = \sum_{j=0}^{n} \binom{n}{j} e_j(I) r^j s^{n-j} + O((r+s)^{n-1}).$$

The $e_j(I)$ are the *mixed multiplicities* of I and $\mathfrak{m}$.

Alternatively, for general $h_{j+1}, \ldots, h_n \in R_1$:

$$e_j(I) = e\left(\frac{I + (h_{j+1}, \ldots, h_n)}{(h_{j+1}, \ldots, h_n)}\right).$$

- $e_0(I) = 1$
- $e_1(I) = \operatorname{ord}_{\mathfrak{m}}(I)$
- $e_n(I) = e(I)$
- $e_j(I) = e_j(\bar{I})$

The main result of [5] is the following lower bound for an $\mathfrak{m}$-primary ideal J:

$$\operatorname{lct}(J) \geq \frac{1}{e_1(J)} + \frac{e_1(J)}{e_2(J)} + \cdots + \frac{e_{n-1}(J)}{e_n(J)}. \qquad (1)$$

Let $\operatorname{DP}(J)$ denote the RHS of (1).

Corollary 3.11 [1]

If char $k = p > 0$ and J is $\mathfrak{m}$-primary, then $\operatorname{fpt}(J) \geq \operatorname{DP}(J)$.

Main Theorem 4.14 [1]

If $c(I) = \operatorname{DP}(I)$, then $e_{j+1}(I)/e_j(I) \in \mathbb{Z}^+$ for $0 \leq j \leq n-1$. Moreover, up to linear change of coordinates and integral closure, we have

$$I = \left(x_1^{e_1(I)}, x_2^{e_2(I)/e_1(I)}, \ldots, x_n^{e_n(I)/e_{n-1}(I)}\right).$$

Proof of Main Theorem 4.14

- Write $I =: I_1 + \ldots I_r$, where I_j generated by d_j-forms and $d_1 < \cdots < d_r$.
- Study $I|_L$ for general linear spaces $L \subseteq \mathbb{A}_k^n$ of varying codimension to control the ideals I_j, and induct on r with base cases $r = 1, 2$.
- $\boldsymbol{r = 1}$**:** [4, Theorem 1.4] or [6, Proposition 4.5].
- $\boldsymbol{r = 2}$**:** In char 0, follows from [3, Theorem 3.5]. In char $p > 0$, a new argument is needed.

Theorem A [2]

Let char $k > 0$, and let $J \subseteq R$ be an ideal generated by d-forms. Then $\operatorname{fpt}(J) = \operatorname{codim}(J)/d$ if and only if, up to change of coordinates and integral closure, we have $J = (x_1, \ldots, x_{\operatorname{codim}(J)})^d$.

Future Work

- Theorem 4.14 fails for non-homogeneous ideals: consider $I = (x + y^2, y^3) \subseteq k[x, y]$.
- If we extend to $k[\![x, y]\!]$ and allow non-linear changes of coordinates, there is still hope: if $\varphi : k[\![x, y]\!] \to k[\![x, y]\!]$ such that $\varphi(x + y^2) = x$ and $\varphi(y) = y$, then

$$\varphi(\bar{I}) = \overline{(x, y^3)}.$$

Local Conjecture

Let k be an algebraically-closed field and $(R, \mathfrak{m}) = (k[\![x_1, \ldots, x_n]\!], (\underline{x}))$. Let $I \subseteq R$ be $\mathfrak{m}$-primary with $c(I) = \operatorname{DP}(I)$. Then $e_{j+1}(I)/e_j(I) \in \mathbb{Z}^+$ for $0 \leq j \leq n-1$. Moreover, there exists an automorphism $\varphi : R \to R$ with

$$\varphi(\bar{I}) = \overline{(x_1^{e_1(I)}, x_2^{e_2(I)/e_1(I)}, \ldots, x_n^{e_n(I)/e_{n-1}(I)})}.$$

Analytic Question

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain containing 0. Let $\phi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be plurisubharmonic with an isolated singularity at 0. Suppose $c(\phi) = \operatorname{DP}(\phi)$ (see [5] for relevant definitions). Must there exist $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, biholomorphic at 0, such that $\varphi(0) = 0$ and

$$(\phi \circ \varphi)(z) = \left(\log \max_{0 \leq i \leq n-1} \frac{e_{i+1}(\phi)|z_i|}{e_i(\phi)}\right) + O(1)?$$

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Acknowledgements

This research was conducted at the University of Michigan while the author was funded by NSF grant DMS-2101075 and NSF RTG grant DMS-1840234. Thanks to my advisor, Karen Smith.