ON LOWER BOUNDS FOR THE F-PURE THRESHOLDS OF EQUIGENERATED IDEALS

BENJAMIN BAILY

Abstract. Let k be a field of characteristic $p > 0$ and $R = k[x_0, \dots, x_n]$. We consider ideals $I \subseteq R$ generated by d-forms. Takagi and Watanabe proved that $\mathrm{fpt}(I) \geq \mathrm{height}(I)/d$; we classify ideals I for which equality is attained. Additionally, we describe a new relationship between $\mathrm{fpt}(I)$ and $\mathrm{fpt}(I|_H)$, where H is a general hyperplane through the origin. As an application, for degree-d homogeneous polynomials $f \in R$ with $p \geq n - 1$, we show that either p divides the denominator of $\mathrm{fpt}(f)$ or $\mathrm{fpt}(f) \geq r/d$, where r is the codimension of the singular locus of f.

1. Introduction

The F-pure threshold, introduced by Takagi and Watanabe [26], is a numerical singularity invariant of pairs in positive characteristic. The F-pure threshold was proposed as a positive-characteristic analog of the log canonical threshold; whereas the log canonical threshold is widely studied in birational and complex-analytic geometry, the F-pure threshold better reflects the subtleties of singularities in prime characteristic.

We consider a pair (R, I), where R is a polynomial ring over a field and I is generated by homogeneous forms of degree d. In this setting, Takagi and Watanabe proved the following sharp lower bound on the F-pure threshold $\mathrm{fpt}(I)$:

Proposition 1.1 ([26], Proposition 4.2)**.** *Let k be a field of positive characteristic and set $R = k[x_0, \dots, x_n]$. Suppose $I \subseteq R$ is generated by forms of degree d. If h is the height of I, then* $\mathrm{fpt}(I) \geq h/d$.

If we instead consider a field of characteristic 0 and the log canonical threshold (lct), much more is known. We refer the reader to [18] for background on log canonical singularities and the lct.

Theorem 1.2 ([6], Theorem 3.5)**.** *Let k be an algebraically closed field of characteristic zero and set $R = k[x_0, \dots, x_n]$. Suppose $I = (f_1, \dots, f_r) \subseteq R$ is generated by forms of degree d. Let e denote the codimension of Z, where Z is the non-klt locus of $(R, I^{\mathrm{lct}(I)})$. Then we have* $\mathrm{lct}(I) \geq e/d$ *with equality if and only if there exist independent linear forms $\ell_1, \dots, \ell_e \in R$ such that $Z = (\ell_1, \dots, \ell_e)$ and $f_i \in k[\ell_1, \dots, \ell_e]$ for all $1 \leq i \leq r$.*

Our goal is to bridge the gap between Proposition 1.1 and Theorem 1.2. As we show in Example 5.2, a naive translation of Theorem 1.2 into characteristic p is not true without an additional hypothesis. Towards the goal of bridging this gap, we contribute two results. The first is a classification of ideals for which the lower bound in Proposition 1.1 is sharp.

Theorem A. *Let k be an algebraically-closed field of characteristic $p > 0$. Let I be a homogeneous ideal in $k[x_0, \dots, x_n]$ generated by d-forms. If h is the height of I, then* $\mathrm{fpt}(I) = h/d$ *if and only if* $\overline{I} = (x_0, \dots, x_{h-1})^d$ *up to change of coordinates.*

The proof of Theorem A goes as follows. First, we prove the claim in the case that I is complete intersection of height n, see Lemma 3.15. In this case, let $\mathfrak{p}$ be a minimal prime over I. Since $\mathfrak{p}$ is the ideal of a point in $\mathbb{P}^n$, we may change coordinates so that $\mathfrak{p} = (x_1, \dots, x_n)$. We then transform

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Theorem A to a statement about the monomial ideals $\{\text{in}_{>_{\text{lex}}}(I^m)\}_{m>0}$, which we solve using convex geometry. After applying estimates for the Hilbert series of powers of I (Lemma 3.16, the result is a consequence of a 1960 result of Grünbaum, Theorem 3.6.

To generalize beyond the case of a complete intersection, we note that $h := \text{height}\, I$, then any general d-forms in I generate a complete intersection $J \subseteq I$, and we show that J is a reduction of I. To generalize beyond the case that $\text{height}\, I = n$, we consider $I|_H$, where H is a general hyperplane through the origin. By Proposition 1.1, we have $h/d \leq \text{fpt}(I|_H) \leq \text{fpt}(I) = h/d$, so $\overline{I|_H} = (x_0, \dots, x_{h-1})^d$. By Proposition 3.4, we deduce that I has the same form.

Our second contribution is a lower bound $\text{fpt}(R, f)$ in terms of the codimension of the singular locus of f. Compare with [5, Theorem 1.1], a preprint which was later superseded by [6].

Theorem B. *Suppose* char $k = p > 0$ *and* $R = k[x_0, \dots, x_n]$. *Suppose that* $f \in R$ *is homogeneous of degree* d, *that* f *is smooth in codimension* c, *and that* p *does not divide the denominator of* $\text{fpt}(f)$. *Further suppose that* $c \geq n$ *or* $p \geq c$. *Then* $\text{fpt}(f) \geq \min(c/d, 1)$.

In the case that $h \geq n$, we observe that $(R, f^{\text{fpt}(f)})$ has an F-pure center which is a monomial ideal and apply a Fedder-type criterion from [23], see Lemma 5.3. When $h < n$, we reduce to the case $h = n$ by intersecting with a general hyperplane through the origin and applying the following Bertini theorem for F-purity:

Theorem C. *Let* k *be an infinite field of characteristic* $p > 0$. *Let* $R = k[x_0, \dots, x_n]$. *Let* $I \subseteq R$ *be an ideal generated by forms of degree at most* d. *Let* $H \in (\mathbb{P}^n)^\vee$ *be a general hyperplane through the origin. Then for all* $0 \leq t < \frac{n}{d} - \frac{n-1}{pd}$, *the pair* (R, I^t) *is sharply* F*-split if and only if* $(H, I^t|_H)$ *is sharply* F*-split.*

A Bertini theorem for F-purity of pairs is already known [25, Theorem 6.1]. Schwede and Zhang's result, however, considers a general member of a free linear system, whereas Theorem C considers a general member of a linear system with $0 \in \mathbb{A}^{n+1}$ as a base point. To ensure that neither [25, Theorem 6.1] nor Theorem C implies the other, we demonstrate in Example 4.4 that the exponent $\frac{n}{d} - \frac{n-1}{pd}$ is optimal.

2. Preliminaries

2.1. The F-Pure Threshold.

For detailed background on the F-pure threshold, we direct the reader to [24, 26]. In this subsection, we summarize several key definitions and results.

Definition 2.1. Let R be a ring of characteristic $p > 0$. We let F_*R denote the R-module structure on R given by restriction of scalars along the Frobenius map $F : R \to R$. We say R is F-finite if F_*R is module-finite over R.

Definition 2.2 ([24])**.** Let R be an F-finite ring, $I \subseteq R$ an ideal, and $t \in \mathbb{R}^+$. The pair (R, I^t) is *sharply* F*-split* if for some (equivalently, infinitely many) $e > 0$, the map

$$I^{\lceil t(p^e-1) \rceil} \cdot \text{Hom}(F^e_* R, R) \to R$$

is surjective.

Definition 2.3 ([26])**.** The F*-pure threshold* of the pair (R, I) is the supremum of all t such that (R, I^t) is sharply F-split. We denote this quantity by $\text{fpt}(R, I)$, or $\text{fpt}(I)$ when the ambient ring is clear.

In practice, the following proposition is a more useful characterization of the F-pure threshold.

Proposition 2.4. *Let* $(R, \mathfrak{m})$ *be an* F*-finite regular local ring. Then the* F*-pure threshold of the pair* (R, I^t) *is equal to*

$$\sup \left\{ \frac{\nu}{p^e} : I^\nu \notin \mathfrak{m}^{[p^e]} \right\}.$$

In fact, let $\nu_I(p^e) = \max\{r : I^r \notin \mathfrak{m}^{[p^e]}\}$. *Then the* F*-pure threshold of* $(R, \mathfrak{a})$ *is equal to the limit* $\lim_{e\to\infty} \nu_I(p^e)/p^e$. *If instead* R *is a polynomial ring over an* F*-finite field and* $I \subseteq R$ *a homogeneous ideal, then the same results hold when we let* $\mathfrak{m}$ *denote the homogeneous maximal ideal of* R.

Proof. The first claim follows from [26, Lemma 3.9]. The existence of the limit is [19, Lemma 1.1]. For the graded setting, see [3, Proposition 3.10]. ☐

Proposition 2.5 (Properties of the F-pure threshold)**.** *Let* R *be a reduced,* F*-finite,* F*-pure ring of characteristic* $p > 0$. *Then for all ideals* $I \subseteq R$ *such that* I *contains a nonzerodivisor, we have*

(i) *If* $I \subseteq J$, *then* $\mathrm{fpt}(I) \leq \mathrm{fpt}(J)$.
(ii) *For all* $m > 0$, *we have* $\mathrm{fpt}(I^m) = m^{-1}\,\mathrm{fpt}(I)$.
(iii) *We have* $\mathrm{fpt}(I) = \mathrm{fpt}(\overline{I})$, *where* $\overline{I}$ *denotes the integral closure of* I.

Proof. See [26, Proposition 2.2] (1), (2), (6). ☐

Proposition 2.6. *Let* $R = k[x_0, \dots, x_n]$. *Let* $>$ *be a monomial order. Let* $I \subseteq R$ *be an ideal, and* $\mathrm{in}_>(I)$ *the initial ideal of* I *with respect to* $>$. *Then* $\mathrm{fpt}(\mathrm{in}_>(I)) \leq \mathrm{fpt}(I)$.

Proof. See [26], the claim preceding Remark 4.6. ☐

2.2. **Newton Polytopes of Monomial Ideals.** When working with monomial ideals, one often identifies a monomial $x_0^{a_0} \cdots x_n^{a_n}$ with the point $(a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. For future reference, it will help to give a name to this identification.

Definition 2.7. Let k be a field. We define the map

$$\log : \{\text{monomials in } k[x_0, \dots, x_n]\} \to \mathbb{Z}_{\geq 0}^{n+1}, \qquad \log(x_0^{a_0} \cdots x_n^{a_n}) = (a_0, \dots, a_n).$$

Definition 2.8. Let $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ be a monomial ideal. Then the *Newton Polytope* of I, denoted $\Gamma(\mathfrak{a})$, is the convex hull in $\mathbb{R}^{n+1}$ of $\log(\mathfrak{a})$. Later on, we will let $\mathrm{conv}(-)$ denote the convex hull of a set.

Remark 2.9. We record several properties of $\Gamma(\mathfrak{a})$.

(i) $\Gamma(\mathfrak{a})$ is a closed, convex, unbounded subset of the first orthant of $\mathbb{R}^n$.
(ii) When $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the complement of $\Gamma(\mathfrak{a})$ inside the first orthant is an open, bounded polyhedron.
(iii) For two ideals $\mathfrak{a}, \mathfrak{b}$, the Minkowski sum of $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{b})$ is equal to $\Gamma(\mathfrak{a}\mathfrak{b})$. In particular, $\Gamma(\mathfrak{a}^n) = n\Gamma(\mathfrak{a})$.

For the proof of Theorem A, we will also require the following.

Definition 2.10. We define the standard n-simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$ as follows:

$$\Delta_n = \{(a_0, \dots, a_n) : 0 \leq a_i, a_0 + \cdots + a_n = 1.\}$$

Definition 2.11. Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal and $t \in \mathbb{Z}^+$. We let $[I]_t$ denote the vector space of t-forms in I.

Definition 2.12. Let $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ be a monomial ideal and $t \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}, t)$ as the convex hull of $\log([\mathfrak{a}]_t)$, and we let $\gamma(\mathfrak{a}, t)$ denote the relative interior of $\Gamma(\mathfrak{a}, t)$ inside $t\Delta_n$.

Remark 2.13. It is sometimes the case that $\Gamma(\mathfrak{a}, t) \subsetneq \Gamma(\mathfrak{a}) \cap t\Delta_n$, even if $\mathfrak{a}$ is integrally closed. Consider $\mathfrak{a} = (x_0, x_1^3)$ as an ideal of $k[x_0, x_1]$; we have $(0.5, 1.5) \in (\Gamma(\mathfrak{a}) \cap 2\Delta_1) \setminus \Gamma(\mathfrak{a}, 2)$.

The following proposition shows that Newton polytope of a monomial ideal determines the F-pure threshold.

Proposition 2.14 ([15], Proposition 36)**.** *Let* $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ *be a monomial ideal. Then*

$$\mathrm{fpt}(\mathfrak{a}) = \frac{1}{\mu}, \text{ where } \mu = \inf\{t : t\vec{1} \in \Gamma(\mathfrak{a})\}.$$

Following the proof of [7], Theorem 1.4 and the terminology of [17], we also define the *limiting polytope* of a graded system of monomial ideals.

Definition 2.15. Let $\mathfrak{a}_\bullet$ be a graded system of monomial ideals. That is, suppose $\mathfrak{a}_r\mathfrak{a}_s \subseteq \mathfrak{a}_{r+s}$ for all $r, s \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}_\bullet)$ as the closure in $\mathbb{R}^{n+1}$ of the ascending union $\{\frac{1}{2^m}\Gamma(\mathfrak{a}_{2^m})\}_{m>0}$.

Definition 2.16. Let $>$ be a monomial order on R. We set $\Gamma_>(I) = \Gamma(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_n = \mathrm{in}_>(I^n)$.

2.3. Essential Codimension.

Definition 2.17 (Essential Codimension)**.** Let $J \subseteq R = k[x_0, \dots, x_n]$ be a homogeneous ideal. The essential codimension $\mathrm{ess}(J)$ is equal to the minimal r for which there exist linear forms $\ell_1, \dots, \ell_r$ such that J is extended from $I \subseteq k[\ell_1, \dots, \ell_r]$.

Lemma 2.18. *Let $I, J, \ell_1, \dots, \ell_r$ be as in Definition 2.17. Then $ess(I) = r$.*

Proof. The bound $\mathrm{ess}(I) \le r$ is immediate. Conversely, if I is extended from an ideal $I' \subseteq k[\ell'_1, \dots, \ell'_s] \subseteq k[\ell_1, \dots, \ell_r]$, then J is extended from the same ideal, so $\mathrm{ess}(J) \le \mathrm{ess}(I)$. ☐

3. Classification of Minimal F-Pure Thresholds

3.1. A Bertini Theorem for Essential Codimension.

Convention 3.1. We identify $(\mathbb{P}^n)^\vee$ with the space of hyperplanes passing through $0 \in \mathbb{A}^{n+1}$, as opposed to the usual convention of identifying $(\mathbb{P}^n)^\vee$ with the space of hyperplanes in $\mathbb{P}^n$.

The following standard lemma relates the condition $\mathrm{ess}(J) < n+1$ to a more familiar condition.

Lemma 3.2. *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ an ideal generated by d-forms $f_1, \dots, f_r$. Then $ess(J) \le n$ if and only if there exists $p \in \mathbb{P}^n$ such that $J \subseteq \mathfrak{m}_p^d$.*

Proof. If $\mathrm{ess}(J) \le n$, then there exist $\ell_1, \dots, \ell_n \in R_1$ such that J is extended from $k[\ell_1, \dots, \ell_n]$. Setting $p = [(\ell_1, \dots, \ell_n)]$, we have $J \subseteq \mathfrak{m}_p^d$. Conversely, suppose $p \in \mathbb{P}^n$ such that $J \subseteq \mathfrak{m}_p^d$ and change coordinates so that $\mathfrak{m}_p = (x_1, \dots, x_n)$. In this case, no monomial summand of the f_i involves x_0, so J is extended from $k[x_1, \dots, x_n]$. ☐

Lemma 3.3. *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a nonzero ideal generated by d-forms $f_1, \dots, f_r$. Suppose $ess(J) = n+1$. Then for general $H \in (\mathbb{P}^n)^\vee$, we have $ess(J|_H) = n$.*

Proof. Set $Z = \mathrm{Proj}(R/J) \subseteq \mathbb{P}^n$. We define an incidence correspondence as follows:

$$B = \{(z, H) \in Z \times (\mathbb{P}^n)^\vee : z \in H, f_i|_H \in \mathfrak{m}_z^d \text{ for all } 1 \le i \le r\}.$$

Let $p : B \to Z, q : B \to (\mathbb{P}^n)^\vee$ be the projections. Fix $z \in Z$ and change coordinates so that $z = [0 : \cdots : 0 : 1]$. Write $f_i =: g_i + x_n h_i$ for $g_i \in \mathfrak{m}_z^d, h_i \in \mathfrak{m}^{d-1}$. Let $(z, H) \in B_z$ where $H = V(\ell)$. Then there exist $g'_i \in \mathfrak{m}_z^d, h'_i \in \mathfrak{m}^{d-1}$ such that $g_i + x_n h_i = g'_i + \ell h'_i$. Write $h'_i =: g''_i + x_n h''_i$, where $g''_i \in \mathfrak{m}_z^{d-1}$. Then $x_n(h_i - \ell h''_i) = g'_i + \ell g''_i - g_i \in \mathfrak{m}_z^d$, so $h_i - \ell h''_i = 0$. In particular, $\ell \mid h_i$. It follows that $B_z = \{(z, V(\ell) : \ell \mid h_i \text{ for all } 1 \le i \le r\}$. By assumption, $\mathrm{ess}(J) = n+1$. Since $\mathrm{ess}(J) = n+1$, by Lemma 3.2 we have $h_i \ne 0$ for some i. As h_i has at most $d-1$ linear factors, we must have $|B_z| \le d$.

By the previous paragraph, every closed fiber B_z is zero-dimensional, so $\dim B \le \dim Z$. Consequently, $\dim q(B) \le \dim B \le \dim Z < n$, so $q(B)$ is a proper closed subset of $(\mathbb{P}^n)^\vee$, and so for general $H \in (\mathbb{P}^n)^\vee$, there is no $z \in Z$ such that $(z, H) \in B$. Consequently, there is no $z \in Z \cap H$ such that $f_i \in \mathfrak{m}_z^d|_H$ for all i, so another application of Lemma 3.2 gives $\mathrm{ess}(J|_H) = n$. ☐

The following proposition describes the behavior of essential codimension under restriction to a general linear subspace through the origin.

Proposition 3.4. *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a homogeneous ideal. Set $r = \text{height}(J)$. Let $L = (\ell_{r+1}, \dots, \ell_n)$, where the ℓ_i are chosen generally. For $r \le t \le n$, set $L_t = (\ell_{t+1}, \dots, \ell_n)$ and $J_t = \frac{J+L_t}{L_t}$. Then for all $r \le t \le n$, we have $ess(J_t) = \max(t+1, ess(J))$.*

Proof. By induction, it suffices to consider the case $t = n-1$. The case $\text{ess}(J) = n+1$ is covered by Lemma 3.3; it remains to show that $\text{ess}(J_{n-1}) = \text{ess}(J)$ provided $\text{ess}(J) \le n$. Set $s = \text{ess}(J)$ and change coordinates so that J is extended from an ideal $I \subseteq k[x_0, \dots, x_{s-1}]$. Suppose $s \le n$. Let $I' = Ik[x_0, \dots, x_{n-1}]$. By Lemma 2.18, we have $\text{ess}(I') = \text{ess}(I) = \text{ess}(J)$. The isomorphism $k[x_0, \dots, x_n]/(\ell_n) \cong k[x_0, \dots, x_{n-1}]$ identifies J_{n-1} with I', so $\text{ess}(J_{n-1}) = \text{ess}(I') = \text{ess}(J)$. □

3.2. **An Application of Grünbaum's Inequality.**

Definition 3.5. Let $K \subseteq \mathbb{R}^n$ be a compact set with $\text{vol}(K) > 0$. The *centroid* c of K is the arithmetic mean of the points of K, that is, we have

$$c = \left(\int_{y \in K} dy\right)^{-1} \int_{y \in K} y dy.$$

We first recall Grünbaum's inequality, for which we state an equivalent version below.

Theorem 3.6 ([9], Theorem 2)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body and let c denote the centroid of K. Let H^+ be a half-space whose boundary hyperplane H contains c. Then*

$$\text{vol}(H^+ \cap K) \le \left(1 - \left(\frac{n}{n+1}\right)^n\right) \text{vol}(K).$$

Definition 3.7. We let $\mathcal{M}_n$ denote the quantity $\left(1 - \left(\frac{n}{n+1}\right)^n\right)$ from the theorem.

For our purposes, we must characterize the equality case of Theorem 3.6.

Proposition 3.8. *Suppose H^+, H, K are as in Theorem 3.6, with $\text{vol}(K) > 0$ and $\text{vol}(H^+ \cap K) = \mathcal{M}_n \text{vol}(K)$. Let H denote the boundary hyperplane of H^+. Then there exists a convex body $K' \subseteq H^+ \cap K$ and a point $q \in K \setminus H^+$ such that K' is contained in a hyperplane parallel to H and $K = \text{conv}(K' \cup \{q\})$.*

Proof. Follows from [20], Corollary 8. □

Definition 3.9. Let $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the projection onto the first n coordinates and let $T_n := \pi_n(\Delta_n)$.

Lemma 3.10. *Let $T_n := \pi_n(\Delta_n)$. Let $z_n = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ denote the centroid of T_n. Let H^+ be a half-space whose boundary hyperplane H contains z_n. Then*

$$\text{vol}(H^+ \cap T_n) \le \frac{\mathcal{M}_n}{n!}$$

with equality if and only if H is parallel to a facet F of T_n with $F \subseteq H^+$.

Proof. If K' is an $n-1$-dimensional convex set and q a point not contained in the hyperplane supporting K' such that $\text{conv}(K' \cup \{q\})$ is a polytope, then K' is a facet of $\text{conv}(K' \cup \{q\})$. The result therefore follows from Proposition 3.8. □

We recall the following standard fact from convex analysis:

Lemma 3.11 ([21], Corollary 11.6.1)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex set and $x \in \partial K$. Then there exists a half-space H^+ such that $K \subseteq H^+$ and such that $x \in \partial H^+$.*

Lemma 3.12. *Let $P \subseteq T_n$ be a closed convex set with $z_n \notin intP$. Then $\text{vol}(P) \le \mathcal{M}_n/n!$ with equality if and only if P is the intersection of T_n with a half-space satisfying the conditions of Lemma 3.10.*

Proof. If $z_n \notin \partial P$, then for $0 < \varepsilon \leq \operatorname{dist}(P, z_n)$, the set $\{x \in T_n : \operatorname{dist}(x, P) < \varepsilon\}$ is a strictly larger convex set which does not contain z_n in its interior. We may therefore assume $z_n \in \partial P$. By Lemma 3.11, we may replace P by $H^+ \cap T_n$, where H^+ is the half-space containing P with $z_n \in \partial H^+$. In this case, the result is immediate from Lemma 3.10. □

3.3. **Proof of Theorem A.**

To start, we recall a theorem of Rees.

Theorem 3.13 ([16], Proposition 11.2.1, Theorem 11.3.1)**.** *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $I \subseteq J$ two $\mathfrak{m}$-primary ideals. Then $e(I) = I(J)$ if and only if $\overline{I} = \overline{J}$.*

Moreover, the conclusion of Theorem 3.13 holds when R is a polynomial ring and I, J two $\mathfrak{m}$-primary ideals. As a consequence, we may restate the conclusion of Theorem A in terms of essential dimension.

Lemma 3.14. *Let $R = k[x_0, \dots, x_n]$ and $I \subseteq R$ an ideal generated by d-forms. If* $\operatorname{height}(I) = h$, *then the following are equivalent:*

(i) $ess(I) = h$
(ii) $\overline{I} = (x_0, \dots, x_{h-1})^d$ *up to change of coordinates.*
(iii) $I \subseteq (x_0, \dots, x_{h-1})^d$ *up to change of coordinates.*

Proof.

(i) $\Longrightarrow$ (ii): Up to change of coordinates, I is extended from an ideal $I' \subseteq k[x_0, \dots, x_{h-1}] =: R'$. By flatness of $R' \hookrightarrow R$, we have $\operatorname{height}(I) = h$. There exists an ideal $J' \subseteq I'$ generated by a subset of the generators of I' such that J' is a $(d, \dots, d)$-complete intersection of height h. Let $\mathfrak{m}'$ denote the homogeneous maximal ideal of R'. We have $e(J') = d^h = e((x_0, \dots, x_{h-1})^d)$, so $\overline{J'} = \overline{(\mathfrak{m}')^d}$ by Theorem 3.13. As $(\mathfrak{m}')^d R$ is integrally closed and $I \subseteq (\mathfrak{m}')^d R$, we conclude $\overline{I} = (\mathfrak{m}')^d R$.

(ii) $\Longrightarrow$ (iii): This follows from the containment $I \subseteq \overline{I}$.

(iii) $\Longrightarrow$ (i): This follows from the argument of Lemma 3.2.

□

We state the main technical lemma of this section.

Lemma 3.15. *Let k be an algebraically-closed field of characteristic $p > 0$ and let $R = k[x_0, \dots, x_n]$. Let $I = (f_1, \dots, f_n) \subseteq R$ denote a complete intersection ideal generated by d-forms. Then* $\operatorname{fpt}(I) \geq n/d$, *with equality if and only if $ess(I) = n$.*

We begin with a computation of the Hilbert series of R/I^s.

Lemma 3.16. *Let I, R be as in Lemma 3.15. For $t \geq (d-1)n + d(s-1)$, we have $H_R(R/I^s, t) = \binom{n+s-1}{n} d^n$. In particular, this holds for $t \geq d(s+n)$.*

Proof. We define

$$\mathcal{L}_{n,s} := \{(a_1, \dots, a_n) : a_i \geq 0, a_1 + \dots + a_n \leq s - 1\}.$$

By [10], Corollary 2.3, we have

$$H_R(R/I^s, t) = \sum_{(a_1, \dots, a_r) \in \mathcal{L}_{n,s}} H_R(R/I, t - d(a_1 + \dots + a_n)). \tag{1}$$

The Hilbert series of R/I is given by

$$\sum_{i \geq 0} H_R(R/I, i) t^i = \frac{(1-t^d)^n}{(1-t)^{n+1}} = (1 + t + \dots + t^{d-1})^n (1 + t + t^2 + t^3 + \dots), \tag{2}$$

hence $H_R(R/I,t) = d^n$ for $t \geq (d-1)n$. By Equations (1) and (2), for $t \geq (d-1)n + d(s-1)$ we have

$$H_R(R/I^s,t) = |\mathcal{L}_{n,s}|d^n = \binom{n+s-1}{n} d^n.$$

□

Lemma 3.17. *Let $\mathfrak{a} \subseteq R$ be a monomial ideal containing a monomial m of degree t. For any $t' > t$, if $\frac{t'}{n+1}\vec{1} \in \gamma(\mathfrak{a}, t')$, then $\operatorname{fpt}(\mathfrak{a}) > \frac{n+1}{t'}$.*

Proof. Set $y = \log(m)$. By convexity of $\Gamma(\mathfrak{a})$, we have $\lambda y + (1-\lambda)\gamma(\mathfrak{a}, t') \subseteq \Gamma(\mathfrak{a})$ for all $\lambda \in [0,1]$. Taking $0 < \lambda \ll 1$, we obtain $\frac{\lambda t + (1-\lambda)t'}{n+1}\vec{1} \in \Gamma(\mathfrak{a})$, which implies $\operatorname{fpt}(\mathfrak{a}) \geq \frac{n+1}{\lambda t + (1-\lambda)t'} > \frac{n+1}{t'}$ by Proposition 2.14. □

Lastly, we need a result relating volume and integer point counts for convex bodies.

Lemma 3.18. *Let Δ_n, T_n, π_n be as in Definitions 2.10 and 3.9. For $t \in \mathbb{Z}^+$ and $P \subseteq t\Delta_n$ a convex set, we have*

$$|\operatorname{vol}_n(\pi_n(P)) - \#(P \cap \mathbb{Z}^{n+1})| \leq \sum_{i=1}^{n-1} \frac{t^i}{i!}. \tag{3}$$

Proof. Since $t\Delta_n$ is contained in the affine space $x_0 + \cdots + x_n = t$ and $t \in \mathbb{Z}^+$, π_n induces a bijection between $t\Delta_n \cap \mathbb{Z}^{n+1}$ and $T_n \cap \mathbb{Z}^n$, so Equation (3) can be interpreted as a statement relating the volume and integer point count of $\pi_n(P)$. For each P' occurring as an i-dimensional projection of $\pi_n(P)$ onto an i-dimensional coordinate axis, $\frac{1}{t}P'$ is contained in an i-dimensional simplex, so we have $\operatorname{vol}_i(P') \leq \frac{t^i}{i!}$. The result then follows from [2]. □

We now prove Lemma 3.15.

Proof. Let $\mathfrak{p}$ be a minimal prime over I. Since $k = \overline{k}$ and I is homogeneous, we may change coordinates so that $\mathfrak{p} = (x_1, \ldots, x_n)$. Let $>$ denote the lexicographic order, and define the graded system of ideals $\mathfrak{a}_\bullet = \{\operatorname{in}_>(I^{nm})\}_m$. Since $\mathfrak{p}^r$ is a monomial ideal for all $r \geq 0$ and $I^r \subseteq \mathfrak{p}^r$, we have $\mathfrak{a}_m \subseteq \mathfrak{p}^{nm}$ for all $m \geq 0$. Since $\mathfrak{a}_\bullet$ is graded, we have for any $t \in \mathbb{Z}^+$

$$[\mathfrak{a}_{2^m}]_{2^m t}[\mathfrak{a}_{2^m}]_{2^m t} \subseteq [\mathfrak{a}_{2^m}\mathfrak{a}_{2^m}]_{2^{m+1}t} \subseteq [\mathfrak{a}_{2^{m+1}}]_{2^{m+1}t}.$$

It follows that $\{\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m t)\}_m$ is an ascending chain of convex subsets of H_t. We then set $t = d(n+1)$ and let $\mathcal{P}$ denote the ascending union $\bigcup_{m \geq 1} \gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. If $d\vec{1} \in \mathcal{P}$, there exists some m such that $d\vec{1} \in \gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. By Lemma 3.17, we have $\operatorname{fpt}(\mathfrak{a}_{2^m}) > \frac{n+1}{2^m d(n+1)} = \frac{1}{2^m d}$, so $\operatorname{fpt}(I) > n/d$.

Conversely, suppose $d\vec{1} \notin \mathcal{P}$. Then for all m, we have $d\vec{1} \notin \frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. By Lemma 3.12, we have

$$\operatorname{vol}(\mathcal{P}) = \lim_{m \to \infty} \operatorname{vol}\left(\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))\right) \leq (d(n+1))^n \frac{\mathcal{M}_n}{n!}. \tag{4}$$

We now derive a lower bound for $\operatorname{vol}(\mathcal{P})$. First, by Lemma 3.16, we have

$$\begin{aligned}
\#\mathbb{Z}^{n+1} \cap (\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))) &\geq H_R(\mathfrak{a}_{2^m}, 2^m d(n+1)) \\
&= H_R(I^{2^m n}, 2^m d(n+1)) \\
&= \binom{n + 2^m d(n+1)}{n} - \binom{n + 2^m n - 1}{n} d^n
\end{aligned}$$

provided $2^m d(n+1) \geq d(2^m n + n)$, which is satisfied for all $m \geq \log_2 n$. Using the approximation $\binom{a+b}{b} = \frac{a^b}{b!} + O_b(a^{b-1})$, we have

$$\#\mathbb{Z}^{n+1} \cap (\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))) \geq \frac{(2^m d)^n}{n!}\left((n+1)^n - n^n\right) + O(2^{m(n-1)}) \tag{5}$$

Combining the bounds Lemma 3.18 and eq. (5), we have

$$\begin{aligned}
\operatorname{vol}(\mathcal{P}) &= \lim_{m\to\infty} \operatorname{vol}\left(\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))\right) = \lim_{m\to\infty} \frac{1}{2^{mn}} \operatorname{vol}(\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))) \\
&\geq \lim_{m\to\infty} \frac{1}{2^{mn}} \left(\#(\mathbb{Z}^{n+1} \cap \gamma(\mathfrak{a}_{2^m}, 2^m d(n+1)) - \sum_{i=1}^{n-1} \frac{(2^m d(n+1))^i}{i!}\right) \\
&\geq \lim_{m\to\infty} \frac{1}{2^{mn}} \left(\frac{(2^m d)^n}{n!}\left((n+1)^n - n^n\right) + O(2^{m(n-1)})\right) \\
&= (d(n+1))^n \frac{\mathcal{M}_n}{n!}.
\end{aligned}$$

It follows that $\operatorname{vol}(\mathcal{P}) = \operatorname{vol}(\overline{\mathcal{P}}) = (d(n+1))^n \frac{\mathcal{M}_n}{n!}$, so by Lemma 3.10, we have $\overline{\mathcal{P}} = H^+ \cap (d(n+1))\Delta_n$. Moreover, the boundary hyperplane H of H^+ is parallel to a facet F of $(d(n+1))\Delta_n$ with $F \subseteq H^+$ and $d(n+1)\eta_n \in H$.

For $\alpha \in \mathbb{R}$, define $D_{t,\beta} = \{(a_0, \dots, a_n) \in t\Delta_n : a_0 \leq \beta\}$. Since $\mathfrak{a}_m \subseteq \mathfrak{p}^{mn}$, for any monomial $x_0^{a_0} \dots x_n^{a_n} \in (\mathfrak{a}_m)_t$, we have $a_1 + \dots + a_n \geq mn$ and hence $a_0 \leq t - mn$. In particular, for all $m \geq 0$ we have

$$\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1)) \subseteq D_{2^m d(n+1), 2^m n(d+1) - 2^m n}.$$

As a consequence, we conclude $\mathcal{P} \subseteq D_{d(n+1), d(n+1)-n}$. As $F \subseteq \mathcal{P}$, the only possible choice for F is the facet $\{a_0 = 0\} \subseteq d(n+1)\Delta_n$. We conclude that $\overline{\mathcal{P}} = D_{d(n+1),d}$. We then have

$$\Gamma(\mathfrak{a}_1, d(n+1)) \subseteq \overline{\mathcal{P}} = D_{d(n+1),d} = \Gamma(\mathfrak{p}^{nd}, d(n+1)),$$

so $[\mathfrak{a}_1]_{d(n+1)} \subseteq [\mathfrak{p}^{nd}]_{n(d+1)}$. For each generator f_i of I, we have $x_0^d \operatorname{in}_>(f_i^n) \in [\mathfrak{a}_1]_{d(n+1)} \subseteq [\mathfrak{p}^{nd}]_{n(d+1)}$, so $x_0 \nmid \operatorname{in}_>(f_i^n)$ for all i. As $\operatorname{in}_>(f_i^n) = \operatorname{in}_>(f_i)^n$, we deduce that $I \subseteq \mathfrak{p}^d$. By Lemma 3.14, we have $\operatorname{ess}(I) = n$. □

We are now able to prove Theorem A. By Lemma 3.14, it suffices to prove the following.

Theorem (A)**.** *Let k be an algebraically-closed field of characteristic $p > 0$. Let I be a homogeneous ideal in $k[x_0, \dots, x_n]$ generated by d-forms and set $h = \operatorname{height}(I)$. Then $\operatorname{fpt}(I) = h/d$ if and only if ess$(I) = h$.*

Proof. Let k be an algebraically-closed field and $R = k[x_0, \dots, x_n]$. Let $I \subseteq R$ be an ideal generated by d-forms, and suppose that $\operatorname{height}(I) = n, \operatorname{fpt}(I) = n/d$. If $f_1, \dots, f_n$ are n general d-forms in I, then $J = (f_1, \dots, f_n)$ is a complete intersection. By Proposition 1.1 and Proposition 2.5 (i), we have

$$n/d \leq \operatorname{fpt}(J) \leq \operatorname{fpt}(I) = n/d.$$

By Lemmas 3.14 and 3.15, we may change coordinates on R such that $\overline{J} = (x_1, \dots, x_n)^d$. Then we have $(x_1, \dots, x_n)^d \subseteq \overline{I}$. Let $>$ denote the lexicographic order, and let g be a d-form in $\overline{I}$. Write $\operatorname{in}_>(g) = x_0^{a_0} \cdots x_n^{a_n}$, and note that $(x_1, \dots, x_n) \subseteq \operatorname{in}_>(\overline{I})$. Set $a = \max_i a_i$. Then

$$g^{\lfloor (p^e-1)/a \rfloor} \prod_{i=1}^{n} (x_i^d)^{\lfloor ((p^e-1) - a_i \lfloor (p^e-1)/a \rfloor)/d \rfloor} \notin \mathfrak{m}^{[p^e]},$$

so we have

$$\lim_{e\to\infty}\frac{\nu_{\mathrm{in}_>(\overline{I})}}{p^e}\geq\lim_{e\to\infty}\frac{1}{p^e}\left(\left\lfloor\frac{p^e-1}{a}\right\rfloor+\sum_{i=1}^{n}\left\lfloor\frac{p^e-1}{d}-\frac{a_i\lfloor(p^e-1)/a\rfloor}{d}\right\rfloor\right)=\frac{1}{a}+\sum_{i=1}^{n}\left(\frac{1}{d}-\frac{a_i}{ad}\right)=\frac{n}{d}+\frac{a_0}{ad}.$$

Consequently, by Propositions 2.5 and 2.6 we have

$$\frac{n}{d}=\mathrm{fpt}(I)=\mathrm{fpt}(\overline{I})\geq\mathrm{fpt}(\mathrm{in}_>(\overline{I}))\geq\mathrm{fpt}((x_1,\ldots,x_n)^d+(x_0^{a_1}\cdots x_n^{a_n}))=\frac{n}{d}+\frac{a_0}{ad},$$

so we have $a_0=0$, hence $\mathrm{in}_>(g)\in(x_1,\ldots,x_n)^d$. As $>$ is the lexicographic order, it follows that $g\in(x_1,\ldots,x_n)^d$. As g was arbitrary, we conclude that $\overline{I}=(x_1,\ldots,x_n)^d$.

Next, we consider the case that height $I\neq n$. If height $I=n+1$, then $\overline{I}=(x_0,\ldots,x_n)^{n+1}$ by Theorem 3.13. Otherwise, suppose height $I=h\leq n-1$. Let L be an ideal generated by $n-h$ linear forms. Then $\frac{h}{d}\leq\mathrm{fpt}(\frac{I+L}{L})\leq\frac{h}{d}$, so by Lemma 3.15, we have $\mathrm{ess}(\frac{I+L}{L})=h$. By Proposition 3.4, the same holds for I. □

4. A Bertini Theorem for F-Purity of Pairs

In this section, we prove Theorem C.

Lemma 4.1 ([1], Lemma 3.2)**.** *Let $R:=k[x_0,\ldots,x_n],\mathfrak{m}:=(x_0,\ldots,x_n)$. For $e,t\in\mathbb{Z}^+$, we have*

$$(\mathfrak{m}^{[p^e]}:\mathfrak{m}^t)=\begin{cases}R & t\geq(n+1)p^e-n\\ \mathfrak{m}^{[p^e]}+\mathfrak{m}^{(n+1)p^e-n-t} & t<np^e-n+1\end{cases}$$

Lemma 4.2. *Let k be a field of characteristic $p>0$, let $R=k[x_1,\ldots,x_n]$, and $I\subseteq\mathfrak{m}$ a homogeneous ideal. For $H=V(\ell)\in(\mathbb{P}^n)^\vee$, we let $I|_H$ denote the image of I in $R/\ell R$. In this case, we have*

$$\nu_{I|_H}(p^e)\leq\max\{r:I^r\not\subseteq\mathfrak{m}^{[p^e]}+\mathfrak{m}^{n(p^e-1)+1}\},\tag{6}$$

Conversely, if $|k|\geq p^e$, then there exists $H\in(\mathbb{P}^n)^\vee(k)$ such that

$$\nu_{I|_H}(p^e)\geq\max\{r:I^r\not\subseteq\mathfrak{m}^{[p^e]}+\mathfrak{m}^{n(p^e-1)-(n-1)(p^{e-1})+1}\}.\tag{7}$$

If k is infinite, then Equation (7) holds for general $H\in(\mathbb{P}^n)^\vee$.

Proof. Let $\mathfrak{a}_e:=\mathfrak{m}^{[p^e]}+\mathfrak{m}^{n(p^e-1)+1}$, $d_e=n(p^e-1)-(n-1)(p^{e-1})$, and $\mathfrak{b}_e:=\mathfrak{m}^{[p^e]}+\mathfrak{m}^{d_e+1}$. We have $\mathfrak{a}_e|_H=\mathfrak{m}^{[p^e]}|_H$, which proves the bound 6. For Equation (7), suppose $f\in I^r\setminus\mathfrak{b}_e$ is a homogeneous element which necessarily has degree at most d_e. By Lemma 4.1, we have $f\notin(\mathfrak{m}^{[p^e]}:\mathfrak{m}^{d_e-\deg(f)})$. Multiplying f by a generator of $\mathfrak{m}^{d_e-\deg(f)}$, we may assume $f\notin\mathfrak{b}_e$ and $\deg f=d_e$. Write

$$f=\sum_{a_0+\cdots+a_n=d_e}c_{a_0,\ldots,a_n}x_0^{a_0}\cdots x_n^{a_n}.$$

For $\lambda\in k^n$, let H_λ denote the hyperplane cut out by $x_0=\lambda_1x_1+\cdots+\lambda_nx_n$. For $b_1,\ldots,b_n\in\mathbb{Z}^{\geq0}$ such that $b_1+\cdots+b_n=d_e$, define

$$P^f_{b_1,\ldots,b_n}(\lambda):=\sum_{a_0=0}^{d_e}\left(\sum_{\substack{a_i\leq b_i\ \forall 1\leq i\leq n\\ a_1+\cdots+a_n=d_e-a_0}}c_{a_0,\ldots,a_n}\binom{a_0}{b_1-a_1,\ldots,b_n-a_n}\lambda_1^{b_1-a_1}\cdots\lambda_n^{b_n-a_n}\right).$$

Then we have

$$\begin{aligned}f|_{H_\lambda}&=\sum_{a_0+\cdots+a_n=d_e}c_{a_0,\ldots,a_n}x_1^{a_1}\cdots x_n^{a_n}(\lambda_1x_1+\ldots\lambda_nx_n)^{a_0}\\&=\sum_{b_1+\cdots+b_n=d_e}x_1^{b_1}\ldots x_n^{b_n}P^f_{b_1,\ldots,b_n}(\lambda).\end{aligned}$$

For any H_λ, we have that $f|_{H_\lambda} \notin \mathfrak{m}^{[p^e]}$ if and only if there exist $b_1, \ldots, b_n \leq p^e - 1$ for which $P^f_{b_1,\ldots,b_n}(\lambda) \neq 0$. We first prove that there exist $b_1, \ldots, b_n \leq p^e - 1$ for which $P^f_{b_1,\ldots,b_n}(\lambda)$ is a non-constant polynomial in λ. To this end, it suffices to produce $a_0, \ldots, a_n, b_1, \ldots, b_n$ such that

(i) $a_0 + \cdots + a_n = d_e$;
(ii) $c_{a_0,\ldots,a_n} \neq 0$;
(iii) $b_1 + \cdots + b_n = d_e$;
(iv) $a_i \leq b_i \leq p^e - 1$ for all $1 \leq i \leq n$;
(v) We have
$$\binom{a_0}{b_1 - a_1, \ldots, b_n - a_n} \not\equiv 0 \mod p.$$

By assumption that $f \notin \mathfrak{m}^{[p^e]}$, it is possible to choose $a_0, \ldots, a_n$ such that $a_0 + \cdots + a_n = d_e, a_0, \ldots, a_n \leq p^e - 1$, and $c_{a_0,\ldots,a_n} \neq 0$. We will prove, by induction on the p-ary digits of a_0, that there exist $b_1, \ldots, b_n$ satisfying (iii)-(v). The base case, when $a_0 = 0$, is obvious. Write $a_0 = \alpha_0 + \alpha_1 p + \cdots + \alpha_{e-1}p^{e-1}$ and suppose $\alpha_j \neq 0$.

As $a_1 + \cdots + a_n = d_e - a_0$, we have $\min_i a_i \leq \frac{d_e - a_0}{n}$. As a consequence, we have

$$\max_{1\leq i\leq n} (p^e - 1) - a_i \geq (p^e - 1) - \frac{d_e - a_0}{n} \geq (p^e - 1) - \frac{d_e - p^j}{n} = \frac{(n-1)(p^{e-1})}{n} + \frac{p^j}{n} \geq p^j.$$

It follows that there exists some $1 \leq i \leq n$ with $a_i + p^j \leq p^e - 1$. We apply the induction hypothesis to produce integers $b_1, \ldots, b_n$ satisfying (iii)-(v) with respect to $(a_0 - p^j, a_1, \ldots, a_i + p^j, \ldots, a_n)$. Since
$$\binom{a_0 - p_j}{b_1 - a_1, \ldots, b_i - a_i - p_j, \ldots, b_n - a_n} \not\equiv 0 \mod p,$$
it follows by Lucas's theorem that we can perform the addition
$$(b_1 - a_1) + \cdots + (b_i - a_i - p^j) + \cdots + (b_n - a_n) = a_0 - p^j$$
in base p without having to carry a digit. Consequently, the same is true for the addition
$$(b_1 - a_1) + \cdots + (b_i - a_i) + \cdots + (b_n - a_n) = a_0,$$
so $b_1, \ldots, b_n$ satisfy conditions (iii)-(v) for the original tuple $(a_1, \ldots, a_n)$.

For $b_1, \ldots, b_n$ as above, $P^f_{b_1,\ldots,b_n}(\lambda)$ is a nonzero polynomial of total degree $a_n \leq p^e - 1$ in the variables $\lambda_1, \ldots, \lambda_n$. By the Schwartz-Zippel lemma ([22], Corollary 1), there exist $\lambda_1, \ldots, \lambda_n \in k$ for which $P^f_{b_1,\ldots,b_n}(\lambda_1, \ldots, \lambda_n) \neq 0$. If k is infinite, then $P^f_{b_1,\ldots,b_n}(\lambda) \neq 0$ for general $\lambda \in k^n$. □

As a consequence, we have the following.

Theorem (C)**.** *Let k be an infinite field of characteristic $p > 0$. Let $R = k[x_0, \ldots, x_n]$. Let $I \subseteq R$ be an ideal generated by forms of degree at most d. Let $H \in (\mathbb{P}^n)^\vee$ be a general hyperplane through the origin. Then for all $0 \leq t < \frac{n}{d} - \frac{n-1}{pd}$, the pair (R, I^t) is sharply F-split if and only if $(H, (I|_H)^t)$ is sharply F-split.*

Proof. The implication $(H, (I|_H)^t)$ sharply F-split $\implies (R, I^t)$ sharply F-split is well-known and additionally is immediate from Equation (6). Conversely, suppose (R, I^t) is sharply F-split and $t < \frac{n}{d} - \frac{n-1}{pd}$. Then there exists $M \in \mathbb{Z}^+$ such that for all $e \in \mathbb{Z}^+, M \mid e$, we have $I^{\lceil t(p^e-1)\rceil} \not\subseteq \mathfrak{m}^{[p^e]}$. Since $t < \frac{n}{d} - \frac{n-1}{pd}$, we may choose $e \gg 0$ divisible by M such that

$$\lceil t(p^e - 1)\rceil d \leq n(p^e - 1) - (n-1)(p^{e-1}). \tag{8}$$

By assumption we have $I^{\lceil t(p^e-1)\rceil} \not\subseteq \mathfrak{m}^{[p^e]}$, and by Equation (8) none of the generators of $I^{\lceil t(p^e-1)\rceil}$ are contained in $\mathfrak{m}^{n(p^e-1)-(n-1)(p^{e-1})+1}$. By Equation (7), we conclude $I|_H^{\lceil t(p^e-1)\rceil} \not\subseteq \mathfrak{m}^{[p^e]}|_H$, so $(H, (I|_H)^t)$ is not sharply F-split. □

In terms of the F-pure threshold, Theorem C says the following.

Corollary 4.3. *Let R, I, H be as in Theorem C. Then*
$$\min\left(\frac{n}{d} - \frac{n-1}{pd}, \operatorname{fpt}(I)\right) \leq \operatorname{fpt}(I|_H).$$

Proof. By Equation (7), we have
$$\begin{aligned}\operatorname{fpt}(I|_H) &\geq \lim_{e\to\infty} p^{-e} \sup\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{n(p^e-1)-(n-1)p^{e-1}+1}\} \\ &= \lim_{e\to\infty} p^{-e} \min\left(\nu_I(p^e), \left\lfloor \frac{n(p^e-1)-(n-1)p^{e-1}+1}{d} \right\rfloor\right) = \min\left(\operatorname{fpt}(I), \frac{n}{d} - \frac{n-1}{pd}\right).\end{aligned}$$
□

Example 4.4. In Theorem C, our bound on t is optimal. If char $k = p$ and $R = k[x_0, \dots, x_n]$, then we may take $f = x_0(x_1 \cdots x_n)^{p-1}$ and $t = \frac{n}{n(p-1)+1} - \frac{n-1}{p(n(p-1)+1)} = \frac{1}{p}$. Then $\operatorname{fpt}(R, f) = \frac{1}{p-1}$, so (R, f^t) is sharply F-split. For any hyperplane $H \subseteq R$ we have $f|_H \in \mathfrak{m}|_H^{[p]}$, so $\operatorname{fpt}(f|_H) \leq t$. Since p divides the denominator of t, we have that $(H, f^t|_H)$ is not sharply F-split.

Finally, we note that an analog of Corollary 4.3 holds for the log canonical threshold.

Corollary 4.5. *Let k be a characteristic zero field and let R, I, H be as in Theorem C. Then*
$$\min\left(\frac{n}{d}, \operatorname{lct}(I)\right) \leq \operatorname{lct}(I|_H).$$
In particular, if I is generated by d-forms, then the above inequality is an equality.

Proof. Let $\{I_p\}_p$ be a family of positive-characteristic models for I. Using [19], one may prove approximate $\operatorname{fpt}(I_p)$ by $\nu_{I_p}(p)$ to prove a quantitative version of Corollary 4.3 for finite fields. More precisely, one obtains that for each p, there exists $H_p \in (\mathbb{P}^n)^\vee(k_p)$ such that
$$\min\left(\operatorname{fpt}(I), \frac{n(p-1)-(n-1)+1}{pd}\right) + O\left(\frac{1}{p}\right) \leq \operatorname{fpt}(I_p|_{H_p}),$$
where the implicit constant depends on the number of generators of I. By the ACC for log canonical thresholds [8], there exists a hyperplane $H \in (\mathbb{P}^n)^\vee(k)$ such that $\operatorname{lct}(I|_H) \geq \min(\operatorname{lct}(I), \frac{n}{d})$. By semicontinuity of the log canonical threshold [4], the desired bound holds for general $H \in (\mathbb{P}^n)^\vee(k)$.
□

5. The Test Ideal at the Threshold

In the introduction, we claimed that the best-known result in characteristic zero (Theorem 1.2) is stronger than the previous best-known result in positive characteristic (Proposition 1.1). Indeed, Theorem 1.2 shows that analogs of Proposition 1.1 and theorem A holds in characteristic zero.

Proposition 5.1. *Let k be an algebraically-closed field of characteristic zero. Let I be a homogeneous ideal in $k[x_0, \dots, x_n]$ generated by d-forms. If h is the height of I, then $\operatorname{fpt}(I) \geq h/d$ with equality if and only if $\overline{I} = (x_0, \dots, x_{h-1})^d$ up to change of coordinates.*

Proof. Since $(R, (1)^t)$ is klt for all $t > 0$, the non-klt locus Z of $(R, I^{\operatorname{lct}(I)})$ is contained in $V(I)$. Consequently, by Theorem 1.2 we have
$$\operatorname{lct}(I) \geq \frac{\operatorname{codim} Z}{d} \geq \frac{\operatorname{height} I}{d} = \frac{h}{d}.$$
Write $I = (f_1, \dots, f_r)$. If lct $= \frac{h}{d}$, then $\operatorname{codim}(Z) = h$ and $\operatorname{lct}(I) = \frac{\operatorname{codim}(Z)}{d}$, so there exist independent linear forms $\ell_1, \dots, \ell_h \in R$ such that $f_i \in k[\ell_1, \dots, \ell_h]$. Changing coordinates, we may assume $\ell_i = x_{i-1}$ for $1 \leq i \leq h$. The result then follows from Lemma 3.14. □

By [12], the correct positive-characteristic analog of Theorem 1.2 considers strong F-regularity and the F-pure threshold. We direct the reader to [11] for background on the test ideal $\tau(R, \mathfrak{a}^t)$, which cuts out the non-strongly F-regular locus of the pair $(R, \mathfrak{a}^t)$. With this in mind, we are now able to give an example of the failure of Theorem 1.2 in positive characteristic.

Example 5.2. Suppose $p \equiv 2 \mod 3$. Let $R = \mathbb{F}_p[x, y, z]$ and $f = (x^3+y^3+z^3)$. By [13, Theorems 3.1 and 3.3], we have $\operatorname{fpt}(f) = 1-\frac{1}{p}$ and $\tau(R, f^{1-1/p}) = (x, y, z)$. A naive translation of Theorem 1.2 predicts that $\operatorname{fpt}(f) \geq \frac{\operatorname{height}((x,y,z))}{3}$, but this is not the case.

Motivated by the failure of the positive-characteristic analog of Theorem 1.2 in the case that p divides the denominator of $\operatorname{fpt}(f)$, we impose the additional condition that the pair $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split.

Lemma 5.3. *Let k be a field of characteristic $p > 0$. Let $R = k[x_0, \dots, x_n]$. Suppose $I \subseteq R$ is generated by homogeneous polynomials of degree d. Suppose $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split and let $h = \operatorname{height}(\tau(R, I^{\operatorname{fpt}(I)}))$. Suppose further that $h \geq n$. Then $\operatorname{fpt}(I) \geq h/d$.*

Proof. Define the graded system of ideals $\mathfrak{a}_\bullet$ by $\mathfrak{a}_m = I^{\lceil m \operatorname{fpt}(I) \rceil}$. The strongly F-regular loci of $(R, I^{\operatorname{fpt}(I)})$ and $(R, \mathfrak{a}_\bullet)$ coincide according to [23, Definition 2.11]. Let $\mathfrak{p}$ be a minimal prime over $\tau(R, I^{\operatorname{fpt}(I)})$. As $\mathfrak{p}$ is a homogeneous prime ideal of height n or $n+1$, we may change coordinates so that $\mathfrak{p} = (x_0, \dots, x_{h-1})$. By [23], Proposition 4.5 and 4.7, we have that $\mathfrak{p}$ is uniformly $(\mathfrak{a}_\bullet, F)$-compatible, so for all $e \geq 0$ we have $\mathfrak{a}_{p^e-1} \subseteq (\mathfrak{p}^{[p^e]} : \mathfrak{p}) = \mathfrak{p}^{[p^e]} + (x_0 \cdots x_{h-1})^{p^e-1}$. By assumption that $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split, there exists $M > 0$ such that for all $e \geq 0, M \mid e$ we have $\mathfrak{a}_{p^e-1} \not\subseteq \mathfrak{m}^{[p^e]}$. Let $M \mid e$, and let f be a generator of $\mathfrak{a}_{p^e-1}$ such that $f \in \mathfrak{p}^{[p^e]} + (x_0 \cdots x_{h-1})^{p^e-1} \setminus \mathfrak{m}^{[p^e]}$. Then $\lceil \operatorname{fpt}(I)(p^e-1) \rceil d = \deg f \geq h(p^e-1)$, so $\operatorname{fpt}(I) \geq \frac{h}{d}$. ☐

In particular, by [14, Theorem 4.1], the hypothesis that $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split is satisfied whenever I is principal and p does not divide the denominator of $\operatorname{fpt}(I)$.

Theorem (B)**.** *Suppose* $\operatorname{char} k = p > 0$ *and* $R = k[x_0, \dots, x_n]$. *Suppose that $f \in R$ is homogeneous of degree d, that f is smooth in codimension c, and that p does not divide the denominator of* $\operatorname{fpt}(f)$. *Further suppose that $c \geq n$ or $p \geq c$. Then* $\operatorname{fpt}(f) \geq \min(c/d, 1)$.

Proof. We first base change to the algebraic closure of k; this changes neither the hypotheses nor the conclusion. Assume $\operatorname{fpt}(f) < 1$. We will first demonstrate that $V(\tau(R, f^{\operatorname{fpt}(f)})) \subseteq \operatorname{Sing}(f)$. To see this, suppose $\mathfrak{p}$ is a nonsingular point of R/fR. Then in particular, $R_\mathfrak{p}/fR_\mathfrak{p}$ is F-split, so $\operatorname{fpt}(R_\mathfrak{p}, fR_\mathfrak{p}) = 1$ and $\tau(R_\mathfrak{p}, f^{\operatorname{fpt}(f)}R_\mathfrak{p}) = R_\mathfrak{p}$. It follows that $\mathfrak{p} \notin V(\tau(R, f^{\operatorname{fpt}(f)}))$. If $c \in \{n, n+1\}$, by Lemma 5.3 we have $\operatorname{fpt}(f) \geq \operatorname{height}(\tau(R, f^{\operatorname{fpt}(f)}))/d \geq c/d$.

Suppose instead $c \leq n-1$ and $p \geq c$; we'll prove the claim by induction on $n+1-c$. Suppose for the sake of contradiction $\operatorname{fpt}(f) < c/d$. Let H be a general element of $(\mathbb{P}^n)^\vee$. Then $\operatorname{codim}(H, \operatorname{Sing}(f|_H)) = \operatorname{codim}(\operatorname{Spec} R, \operatorname{Sing}(f)) = c$ by Bertini's theorem. As

$$\operatorname{fpt}(f) < \frac{c}{d} \leq \frac{c+1}{d} - \frac{c}{pd} \leq \frac{n}{d} - \frac{c-1}{pd},$$

we have by Corollary 4.3 that $\operatorname{fpt}(f) = \operatorname{fpt}(f|_H)$. In particular, p does not divide the denominator of $\operatorname{fpt}(f)$. By induction, we conclude that $\operatorname{fpt}(f) < c/d \leq \operatorname{fpt}(f|_H) = \operatorname{fpt}(f)$, a contradiction. ☐

We close this paper with a conjecture.

Conjecture 5.4. *Let k be an algebraically-closed field of characteristic $p > 0$ and set $R = k[x_1, \dots, x_n]$. Let $I \subseteq R$ be an ideal generated by d-forms. Further suppose that $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split. Letting $e = \operatorname{height}(\tau(R, I^{\operatorname{fpt}(I)}))$, we have*

$$\operatorname{fpt}(R, I) \geq \frac{e}{d}$$

with equality if and only if $ess(I) = e$.

References

[1] Bhargav Bhatt and Anurag K. Singh. “The F-pure threshold of a Calabi-Yau hypersurface”. In: *Math. Ann.* 362.1-2 (2015), pp. 551–567. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-014-1129-0.

[2] Harold Davenport. “On a principle of Lipschitz”. In: *J. London Math. Soc.* 26 (1951), pp. 179–183. ISSN: 0024-6107,1469-7750. DOI: 10.1112/jlms/s1-26.3.179.

[3] Alessandro De Stefani and Luis Núñez-Betancourt. “F-thresholds of graded rings”. In: *Nagoya Math. J.* 229 (2018), pp. 141–168. ISSN: 0027-7630,2152-6842. DOI: 10.1017/nmj.2016.65.

[4] Jean-Pierre Demailly and János Kollár. “Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds”. In: *Ann. Sci. École Norm. Sup. (4)* 34.4 (2001), pp. 525–556. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(01)01069-2.

[5] Lawrence Ein and Mircea Mustata. *The log canonical threshold of homogeneous affine hypersurfaces*. en. arXiv:math/0105113. May 2001. DOI: 10.48550/arXiv.math/0105113.

[6] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. “Bounds for log canonical thresholds with applications to birational rigidity”. In: *Math. Res. Lett.* 10.2-3 (2003), pp. 219–236. ISSN: 1073-2780. DOI: 10.4310/MRL.2003.v10.n2.a9.

[7] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. “Multiplicities and log canonical threshold”. In: *J. Algebraic Geom.* 13.3 (2004), pp. 603–615. ISSN: 1056-3911,1534-7486. DOI: 10.1090/S1056-3911-04-00346-7.

[8] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. “Shokurov’s ACC conjecture for log canonical thresholds on smooth varieties”. In: *Duke Math. J.* 152.1 (2010), pp. 93–114. ISSN: 0012-7094,1547-7398. DOI: 10.1215/00127094-2010-008.

[9] Branko Grünbaum. “Partitions of mass-distributions and of convex bodies by hyperplanes”. In: *Pacific J. Math.* 10 (1960), pp. 1257–1261. ISSN: 0030-8730,1945-5844.

[10] Elena Guardo and Adam Van Tuyl. “Powers of complete intersections: graded Betti numbers and applications”. In: *Illinois J. Math.* 49.1 (2005), pp. 265–279. ISSN: 0019-2082,1945-6581.

[11] Nobuo Hara and Shunsuke Takagi. “On a generalization of test ideals”. In: *Nagoya Math. J.* 175 (2004), pp. 59–74. ISSN: 0027-7630,2152-6842. DOI: 10.1017/S0027763000008904.

[12] Nobuo Hara and Ken-Ichi Yoshida. “A generalization of tight closure and multiplier ideals”. In: *Trans. Amer. Math. Soc.* 355.8 (2003), pp. 3143–3174. ISSN: 0002-9947,1088-6850. DOI: 10.1090/S0002-9947-03-03285-9.

[13] Daniel J. Hernández. “F-invariants of diagonal hypersurfaces”. In: *Proc. Amer. Math. Soc.* 143.1 (2015), pp. 87–104. ISSN: 0002-9939,1088-6826. DOI: 10.1090/S0002-9939-2014-12260-X.

[14] Daniel J. Hernández. “F-purity of hypersurfaces”. In: *Math. Res. Lett.* 19.2 (2012), pp. 389–401. ISSN: 1073-2780,1945-001X. DOI: 10.4310/MRL.2012.v19.n2.a11.

[15] Daniel J. Hernández. “F-purity versus log canonicity for polynomials”. In: *Nagoya Math. J.* 224.1 (2016), pp. 10–36. ISSN: 0027-7630,2152-6842. DOI: 10.1017/nmj.2016.14.

[16] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*. Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006, pp. xiv+431. ISBN: 978-0-521-68860-4; 0-521-68860-4.

[17] Sarah Mayes. “The limiting shape of the generic initial system of a complete intersection”. In: *Comm. Algebra* 42.5 (2014), pp. 2299–2310. ISSN: 0092-7872,1532-4125. DOI: 10.1080/00927872.2012.758271.

[18] Mircea Mustaţă. “IMPANGA lecture notes on log canonical thresholds”. In: *Contributions to algebraic geometry*. EMS Ser. Congr. Rep. Notes by Tomasz Szemberg. Eur. Math. Soc., Zürich, 2012, pp. 407–442. ISBN: 978-3-03719-114-9. DOI: 10.4171/114-1/16.

[19] Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe. “F-thresholds and Bernstein-Sato polynomials”. In: *European Congress of Mathematics*. Eur. Math. Soc., Zürich, 2005, pp. 341–364. ISBN: 3-03719-009-4.

[20] S. Myroshnychenko, M. Stephen, and N. Zhang. “Grünbaum’s inequality for sections”. In: *J. Funct. Anal.* 275.9 (2018), pp. 2516–2537. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2018.04.001.

[21] R. Tyrrell Rockafellar. *Convex analysis*. Vol. No. 28. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970, pp. xviii+451.

[22] J. T. Schwartz. “Fast probabilistic algorithms for verification of polynomial identities”. In: *J. Assoc. Comput. Mach.* 27.4 (1980), pp. 701–717. ISSN: 0004-5411,1557-735X. DOI: 10.1145/322217.322225.

[23] Karl Schwede. “Centers of F-purity”. In: *Math. Z.* 265.3 (2010), pp. 687–714. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-009-0536-5.

[24] Karl Schwede. “Generalized test ideals, sharp F-purity, and sharp test elements”. In: *Math. Res. Lett.* 15.6 (2008), pp. 1251–1261. ISSN: 1073-2780. DOI: 10.4310/MRL.2008.v15.n6.a14.

[25] Karl Schwede and Wenliang Zhang. “Bertini theorems for F-singularities”. In: *Proc. Lond. Math. Soc. (3)* 107.4 (2013), pp. 851–874. ISSN: 0024-6115,1460-244X. DOI: 10.1112/plms/pdt007.

[26] Shunsuke Takagi and Kei-ichi Watanabe. “On F-pure thresholds”. In: *J. Algebra* 282.1 (2004), pp. 278–297. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2004.07.011.