F-PURE THRESHOLDS OF EQUIGENERATED IDEALS

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ABSTRACT. Let k be a field of characteristic $p > 0$ and $R = k[x_0, \dots, x_n]$. We consider ideals $I \subseteq R$ generated by d-forms. Takagi and Watanabe proved that $\operatorname{fpt}(I) \geq \operatorname{codim}(I)/d$; we classify ideals I for which equality is attained. Additionally, we describe a new relationship between $\operatorname{fpt}(I)$ and $\operatorname{fpt}(I|_H)$, where H is a general hyperplane through the origin. As a consequence, for polynomials $f \in R_d$ with $p \geq n-1$, we show that either p divides the denominator of $\operatorname{fpt}(f)$ or $\operatorname{fpt}(f) \geq r/d$, where r is the codimension of the singular locus of f.

1. INTRODUCTION

The F-pure threshold, introduced by Takagi and Watanabe [18], is an invariant of a pair (R, I), where R is an F-finite F-pure ring of characteristic $p > 0$ and $I \subseteq R$ a proper ideal. The F-pure threshold $\operatorname{fpt}(R, I)$ measures, in a sense, the failure of R/I to be F-pure. We consider the case of an *equigenerated* ideal in a polynomial ring over an algebraically-closed field. Most of the literature on F-pure thresholds of equigenerated ideals [1, 10, 9, 16, 19] concentrates on the case of a principal ideal generated by a homogeneous polynomial; our main result (Theorem 3.17) considers an ideal of arbitrary codimension.

We have the following sharp lower bound on $\operatorname{fpt}(R, I)$:

Proposition 1.1 ([18], Proposition 4.2)**.** *Let k be a field of characteristic $p > 0$ and set $R = k[x_0, \dots, x_n]$. Suppose $I \subseteq R$ is generated by forms of degree d and set $h = \operatorname{codim}(I)$. Then $\operatorname{fpt}(I) \geq h/d$.*

If we instead consider a field of characteristic 0 and the log canonical threshold (lct) of an equigenerated ideal, much more is known.

Theorem 1.2 ([4], Theorem 3.5)**.** *Let k be an algebraically closed field of characteristic zero and set $R = k[x_0, \dots, x_n]$. Suppose $I = (f_1, \dots, f_r) \subseteq R$ is generated by d-forms. Let Z denote the non-klt locus of $(R, I^{\operatorname{lct}(I)})$ and set $e = \operatorname{codim}(Z)$. Then we have $\operatorname{lct}(I) \geq e/d$ with equality if and only if there exist linear forms $\ell_1, \dots, \ell_e$ such that $Z = (\ell_1, \dots, \ell_e)$ and $f_i \in K[\ell_1, \dots, \ell_e]$ for all $1 \leq i \leq r$.*

Our goal is to bridge the gap between Proposition 1.1 and Theorem 1.2. As we show in Example 5.1, a naive translation of Theorem 1.2 into characteristic p is not true without an additional hypothesis. Towards the goal of bridging this gap, we contribute two results. The first is a classification of ideals for which the lower bound in Proposition 1.1 is sharp.

Theorem 3.17. *Let k be an algebraically-closed field of characteristic $p > 0$. Let I be a homogeneous ideal in $k[x_0, \dots, x_n]$ generated by d-forms and set $h = \operatorname{codim}(I)$. Then $\operatorname{fpt}(I) = h/d$ if and only if $\overline{I} = (x_0, \dots, x_{h-1})^d$ up to change of coordinates.*

The proof of Theorem 3.17 goes as follows. First, we prove the claim in the case that I is complete intersection of codimension n, see Lemma 3.14. In this case, let $\mathfrak{p}$ be a minimal prime over I. Since $\mathfrak{p}$ is the ideal of a point in $\mathbb{P}^n$, we may change coordinates so that $\mathfrak{p} = (x_1, \dots, x_n)$. We then transform Theorem 3.17 to a statement about the monomial ideals $\{\operatorname{in}_{>_{\text{lex}}}(I^m)\}_{m>0}$, which we solve using convex geometry. After applying estimates for the Hilbert series of powers of I (Lemma 3.15, the result is a consequence of a 1960 result of Grünbaum, Theorem 3.9.

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To lift the hypothesis that I is a complete intersection, we observe that h general d-forms in I generate a complete intersection $J \subseteq I$, and we show that J is a reduction of I. To lift the hypothesis that $\operatorname{codim} I = n$, we use induction and consider $I|_H$, where H is a general hyperplane through the origin. By Proposition 1.1, we have $h/d \leq \operatorname{fpt}(I|_H) \leq \operatorname{fpt}(I) = h/d$, so $\overline{I|_H} = (x_0, \dots, x_{h-1})^d$. By Proposition 3.3, we deduce that I has the same form.

Our second contribution is a lower bound $\operatorname{fpt}(R, f)$ in terms of the codimension of the singular locus of f. Compare with [3, Theorem 1.1], a preprint which was later superceded by [4].

Theorem 5.3. *Suppose* $\operatorname{char} k = p > 0$ *and* $R = k[x_0, \dots, x_n]$. *Suppose that* $f \in R_d$ *is smooth in codimension* c *and* p *does not divide the denominator of* $\operatorname{fpt}(f)$. *Further suppose that* $c \geq n$ *or* $p \geq c$. *Then* $\operatorname{fpt}(f) \geq \min(c/d, 1)$.

In the case that $h \geq n$, we observe that $(R, f^{\operatorname{fpt}(f)})$ has an F-pure center which is a monomial ideal and apply a Fedder-type criterion from [13], see Lemma 5.2. When $h < n$, we reduce to the case $h = n$ by intersecting with a general hyperplane through the origin and applying the following Bertini theorem for F-purity:

Theorem 4.3. *Let* k *be an infinite field of characteristic* $p > 0$. *Let* $R = k[x_0, \dots, x_n]$. *Let* $I \subseteq R$ *be an ideal generated by forms of degree at most* d. *Let* $H \in (\mathbb{P}^{\mathrm{n}})^\vee$ *be a general hyperplane through the origin. Then for all* $0 \leq t < \frac{n}{d} - \frac{n-1}{pd}$, *the pair* (R, I^t) *is sharply F-split if and only if* $(H, I^t|_H)$ *is sharply F-split.*

A Bertini theorem for F-purity of pairs is already known [15, Theorem 6.1]. Schwede and Zhang's result, however, considers a general member of a free linear system, whereas Theorem 4.3 considers a general member of a linear system with $0 \in \mathbb{A}^{n+1}$ as a base point. To ensure that neither [15, Theorem 6.1] nor Theorem 4.3 implies the other, we demonstrate in Example 4.5 that the exponent $\frac{n}{d} - \frac{n-1}{pd}$ is optimal.

2. Preliminaries

Theorem 2.1 ([17], Proposition 11.2.1, Theorem 11.3.1)**.** *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring and* $I \subseteq J$ *two* $\mathfrak{m}$*-primary ideals. Then* $e(I) = I(J)$ *if and only if* $\overline{I} = \overline{J}$.

2.1. **The F-Pure Threshold.**

Definition 2.2 (F-Pure Threshold, [14] Chapter 4.4)**.** Let R be an F-finite ring, $I \subseteq R$ an ideal, and $t \in \mathbb{R}^+$. The pair (R, I^t) is *sharply F-split* if for some (equivalently, infinitely many) $e > 0$, the map

$$I^{\lceil t(p^e - 1) \rceil} \cdot \operatorname{Hom}(F^e_* R, R) \to R$$

is surjective. The *F-pure threshold* of the pair (R, I) is the supremum of all t such that (R, I^t) is sharply F-split. We denote this quantity by $\operatorname{fpt}(R, I)$, or $\operatorname{fpt}(I)$ when the ambient ring is clear.

In practice, the following proposition is a more useful characterization of the F-pure threshold.

Proposition 2.3 ([14], Exercises 4.19-4.20)**.** *Let* $(R, \mathfrak{m})$ *be an F-finite regular local ring. Then the F-pure threshold of the pair* (R, I^t) *is equal to*

$$\sup \left\{ \frac{\nu}{p^e} : I^\nu \notin \mathfrak{m}^{[p^e]} \right\}.$$

In fact, let $\nu_I(p^e) = \max\{r : I^r \notin \mathfrak{m}^{[p^e]}\}$. *Then the F-pure threshold of* $(R, \mathfrak{a})$ *is equal to the limit* $\lim_{e \to \infty} \nu_I(p^e)/p^e$.

If instead R *is a polynomial ring over an F-finite field and* $I \subseteq R$ *a homogeneous ideal, then the same results hold when we let* $\mathfrak{m}$ *denote the homogeneous maximal ideal of* R.

Proposition 2.4 (Properties of the F-pure threshold)**.** *Let R be a reduced, F-finite, F-pure ring of characteristic $p > 0$. Then for all ideals $I \subseteq R$ such that I contains a nonzerodivisor, we have*

(i) If $I \subseteq J$, then $\operatorname{fpt}(I) \leq \operatorname{fpt}(J)$.
(ii) For all $m > 0$, we have $\operatorname{fpt}(I^m) = m^{-1}\operatorname{fpt}(I)$.
(iii) We have $\operatorname{fpt}(I) = \operatorname{fpt}(\overline{I})$, where $\overline{I}$ denotes the integral closure of I.

Proof. See [18, Proposition 2.2] (1), (2), (6). □

We will require the following essential fact:

Proposition 2.5. *Let $R = k[x_0, \ldots, x_n]$. Let $>$ be a monomial order. Let $I \subseteq R$ be an ideal, and $\operatorname{in}_>(I)$ the initial ideal of I with respect to $>$. Then $\operatorname{fpt}(\operatorname{in}_>(I)) \leq \operatorname{fpt}(I)$.*

Proof. See [18], the claim preceding Remark 4.6. □

2.2. **Newton Polytopes of Monomial Ideals.** When working with monomial ideals, one often identifies a monomial $x_0^{a_0} \cdots x_n^{a_n}$ with the point $(a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. For future reference, it will help to give a name to this identification.

Definition 2.6. Let k be a field. We define the map

$$\log : \{\text{monomials in } k[x_0, \ldots, x_n]\} \to \mathbb{Z}_{\geq 0}^{n+1}, \qquad \log(x_0^{a_0} \cdots x_n^{a_n}) = (a_0, \ldots, a_n).$$

Definition 2.7. Let $\mathfrak{a} \subseteq k[x_0, \ldots, x_n]$ be a monomial ideal. Then the *Newton Polytope* of I, denoted $\Gamma(\mathfrak{a})$, is the convex hull in $\mathbb{R}^{n+1}$ of $\log(\mathfrak{a})$. Later on, we will let $\operatorname{conv}(-)$ denote the convex hull of a set.

Remark 2.8. We record several properties of $\Gamma(\mathfrak{a})$.

(i) $\Gamma(\mathfrak{a})$ is a closed, convex, unbounded subset of the first orthant of $\mathbb{R}^n$.
(ii) When $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the complement of $\Gamma(\mathfrak{a})$ inside the first orthant is an open, bounded polyhedron.
(iii) For two ideals $\mathfrak{a}, \mathfrak{b}$, the Minkowski sum of $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{b})$ is equal to $\Gamma(\mathfrak{a}\mathfrak{b})$. In particular, $\Gamma(\mathfrak{a}^n) = n\Gamma(\mathfrak{a})$.

For the proof of Theorem Theorem 3.17, we will also require the following three conventions.

Definition 2.9. We define the standard n-simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$ as follows:

$$\Delta_n = \{(a_0, \ldots, a_n) : 0 \leq a_i, a_0 + \cdots + a_n = 1.\}$$

Definition 2.10. Let $I \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal and $t \in \mathbb{Z}^+$. We let $[I]_t$ denote the vector space of t-forms in I.

Definition 2.11. Let $\mathfrak{a} \subseteq k[x_0, \ldots, x_n]$ be a monomial ideal and $t \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}, t)$ as the convex hull of $\log([\mathfrak{a}]_t)$, and we let $\gamma(\mathfrak{a}, t)$ denote the relative interior of $\Gamma(\mathfrak{a}, t)$ inside $t\Delta_n$.

Remark 2.12. It is sometimes the case that $\Gamma(\mathfrak{a}, t) \subsetneq \Gamma(\mathfrak{a}) \cap t\Delta_n$, even if $\mathfrak{a}$ is integrally closed. Consider $\mathfrak{a} = (x, y^3)$ as an ideal of $k[x_0, x_1]$; we have $(0.5, 1.5) \in (\Gamma(\mathfrak{a}) \cap 2\Delta_1) \setminus \Gamma(\mathfrak{a}, 2)$.

Proposition 2.13 ([8], Proposition 36)**.** *Let $\mathfrak{a} \subseteq k[x_0, \ldots, x_n]$ be a monomial ideal. Then*

$$\operatorname{fpt}(\mathfrak{a}) = \frac{1}{\mu}, \text{ where } \mu = \inf\{t : t\vec{1} \in \Gamma(\mathfrak{a})\}.$$

Definition 2.14. Let $\mathfrak{a}_\bullet$ be a graded sequence of monomial ideals. That is, suppose $\mathfrak{a}_r\mathfrak{a}_s \subseteq \mathfrak{a}_{r+s}$ for all $r, s \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}_\bullet)$ as the closure in $\mathbb{R}^{n+1}$ of the ascending union $\{\frac{1}{2^m}\Gamma(\mathfrak{a}_{2^m})\}_{m>0}$.

Following the proof of [2], Theorem 1.4 and the terminology of [11], we also define the *limiting polytope* of an ideal $I \subseteq R = k[x_0, \ldots, x_n]$.

Definition 2.15. Let $>$ be a monomial order on R. We set $\Gamma_>(I) = \Gamma(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_n = \operatorname{in}_>(I^n)$.

2.3. Essential Codimension.

Definition 2.16 (Essential Codimension)**.** Let $J \subseteq R = k[x_0, \dots, x_n]$ be a homogeneous ideal. The essential codimension $\mathfrak{e}(J)$ is equal to the minimal r for which there exist linear forms $\ell_1, \dots, \ell_r$ such that J is extended from $I \subseteq k[\ell_1, \dots, \ell_r]$.

Lemma 2.17. *Let $I, J, \ell_1, \dots, \ell_r$ be as in Definition 2.16. Then $\mathfrak{e}(I) = r$.*

Proof. The bound $\mathfrak{e}(I) \leq r$ is immediate. Conversely, if I is extended from an ideal $I' \subseteq k[\ell'_1, \dots, \ell'_s] \subseteq k[\ell_1, \dots, \ell_r]$, then J is extended from the same ideal, so $\mathfrak{e}(J) \leq \mathfrak{e}(I)$. ☐

3. Classification of Minimal F-Pure Thresholds

3.1. A Bertini Theorem for Essential Codimension.

Definition 3.1. We identify $(\mathbb{P}^n)^\vee$ with the space of hyperplanes passing through $0 \in \mathbb{A}^{n+1}$, as opposed to the usual convention of identifying $(\mathbb{P}^n)^\vee$ with the space of hyperplanes in $\mathbb{P}^n$.

While we expect that the following lemma is already known, we could not find the exact statement in the literature.

Lemma 3.2. *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a nonzero homogeneous ideal. Suppose $\mathfrak{e}(J) = n+1$. Then for general $H \in (\mathbb{P}^n)^\vee$, we have $\mathfrak{e}(J|_H) = n$.*

Proof. Write $J = (f_1, \dots, f_s)$ where $\deg f_i = d_i$. Set $Z = \operatorname{Proj}(R/J) \subseteq \mathbb{P}^n$. For a hyperplane $H \in (\mathbb{P}^n)^\vee$, the condition that $\mathfrak{e}(J|_H) < n$ is equivalent to the existence of linear forms $\ell_1, \dots, \ell_n \in R_1$ such that J is extended from $k[\ell_1, \dots, \ell_n]$. Observe that J is extended from $k[\ell_1, \dots, \ell_n]$ if and only if $f_i \in (\ell_1, \dots, \ell_n)^{d_i}$ for all $1 \leq i \leq s$.

We define an incidence correspondence as follows:

$$B = \{(x, H) \in Z \times (\mathbb{P}^n)^\vee : z \in H, f_i|_H \in \mathfrak{m}_x^{d_i} \text{ for all } 1 \leq i \leq s\}.$$

Let $p : B \to Z, q : B \to (\mathbb{P}^n)^\vee$ be the projections. e will prove the claim in the following steps:

1. For each $z \in Z, B_z := p^{-1}(z)$ satisfies $|B_z| < \infty$.
2. Deduce that $q(B)$ is a proper closed subset of $(\mathbb{P}^n)^\vee$.
3. Conclude that a general hyperplane section of Z is not a cone.

Fix $z \in Z$ and change coordinates so that $z = [0 : \cdots : 0 : 1]$. Write $f_i =: g_i + x_n h_i$ for $g_i \in \mathfrak{m}_z^{d_i}, h_i \in \mathfrak{m}^{d_i - 1}$. Let $(z, H) \in B_z$ where $H = V(\ell)$. Then there exist $g'_i \in \mathfrak{m}_z^{d_i}, h'_i \in \mathfrak{m}^{d_i-1}$ such that $g_i + x_n h_i = g'_i + \ell h'_i$. Write $h'_i =: g''_i + x_n h''_i$, where $g''_i \in \mathfrak{m}_z^{d-1}$. Then $x_n(h_i - \ell h''_i) = g'_i + \ell g''_i - g_i \in \mathfrak{m}_z^d$, so $h_i - \ell h''_i = 0$. In particular, $\ell \mid h_i$. It follows that $B_z = \{(z, V(\ell) : \ell \mid h_i \text{ for all } 1 \leq i \leq j\}$. By assumption, $\mathfrak{e}(J) = n+1$. Consequently, we have $h_i \neq 0$ for some i, so $|B_z| \leq \max(d_1, \dots, d_j) < \infty$.

For the second step, every closed fiber B_z is zero-dimensional, so $\dim B \leq \dim Z$. Consequently, $\dim q(B) \leq \dim B \leq \dim Z < n$, so $q(B)$ is a proper closed subset of $(\mathbb{P}^n)^\vee$. For the final step, we note that for general $H \in (\mathbb{P}^n)^\vee$, there is no $z \in Z$ such that $(z, H) \in B$. Consequently, there is no $z \in Z \cap H$ such that $f_i \in \mathfrak{m}_z^{d_i}|_H$ for all i, so $\mathfrak{e}(J|_H) = n$. ☐

The following proposition describes the behavior of essential codimension under restriction to a general linear subspace through the origin.

Proposition 3.3. *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a homogeneous ideal. Set $r = \operatorname{codim}(J)$. Let $L = (\ell_{r+1}, \dots, \ell_n)$, where the ℓ_i are chosen generally. For $r \leq t \leq n$, set $L_t = (\ell_{t+1}, \dots, \ell_n)$ and $J_t = \frac{J + L_t}{L_t}$. Then for all $r \leq t \leq n$, we have $\mathfrak{e}(J_t) = \max(t+1, \mathfrak{e}(J))$.*

Proof. By induction, it suffices to consider the case $t = n-1$. The case $\mathfrak{e}(J) = n+1$ is covered by Lemma 3.2; it remains to show that $\mathfrak{e}(J_{n-1}) = \mathfrak{e}(J)$ provided $\mathfrak{e}(J) \leq n$.

Set $s = \mathfrak{e}(J)$ and change coordinates so that J is extended from an ideal $I \subseteq k[x_0, \dots, x_{s-1}]$. Suppose $s \leq n$. Let $I' = Ik[x_0, \dots, x_{n-1}]$. By Lemma 2.17, we have $\mathfrak{e}(I') = \mathfrak{e}(I) = \mathfrak{e}(J)$. The isomorphism $k[x_0, \dots, x_n]/(\ell_n) \cong k[x_0, \dots, x_{n-1}]$ identifies J_{n-1} with I', so $\mathfrak{e}(J_{n-1}) = \mathfrak{e}(I') = \mathfrak{e}(J)$. $\square$

3.2. **Mixed Volumes and Lattice Points.** To begin, we recall the following theorem of Minkowski.

Theorem 3.4 (Minkowski). *Let $K_1, \dots K_r$ be convex bodies in $\mathbb{R}^n$. Then the function*

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_r K_r)$$

is a homogeneous polynomial of degree n in the variables $\lambda_1, \dots, \lambda_r$.

The coefficients of this polynomial are called the mixed volumes of the convex bodies $K_1, \dots, K_r$.

Definition 3.5. The mixed volumes $V_n(K_1^{\langle a_1 \rangle}, \dots, K_r^{\langle a_r \rangle})$ are defined by the formula

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{a_1 + \cdots + a_r = n} \binom{n}{a_1, \dots, a_r} V_n(K_1^{\langle a_1 \rangle}, \dots, K_r^{\langle a_r \rangle}) \lambda_1^{a_1} \dots \lambda_r^{a_r}.$$

The expression $V_n(K_1^{\langle a_1 \rangle}, \dots, K_r^{\langle a_r \rangle})$ is shorthand for the quantity $V_n(K_1, \dots, K_1, \dots, K_r, \dots, K_r)$ where K_i is repeated a_i times.

Theorem 3.6 (Source). *The coefficients $V_n(K_1, \dots, K_n)$ are well-defined and satisfy the following properties:*

(i) Volume: $V_n(K, \dots, K) = \mathrm{vol}_n(K)$.
(ii) Symmetry: V_n is symmetric in its arguments.
(iii) Multilinearity:

$$V_n(\lambda_1 K_1 + \lambda_2 K_2, K_3, \dots, K_n) = \lambda_1 V_n(K_1, K_3, \dots, K_r) + \lambda_2 V_n(K_2, K_3, \dots, K_n).$$

(iv) Nonnegativity: $V_n(K_1, \dots, K_n) \geq 0$.
(v) Translation-Invariance: $V_n(K_1 + \{x\}, K_2, \dots, K_n) = V_n(K_1, K_2, \dots, K_n)$.
(vi) Monotonicity: If $K_i \subseteq K'_i$ for all i, then $V_n(K_1, K_2, \dots, K_n) \leq V_n(K'_1, K'_2, \dots, K'_n)$.

Moreover, V_n is the unique function satisfying properties (i)-(iii).

Definition 3.7. Let $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the projection onto the first n coordinates.

Lemma 3.8. *Let $\Delta_n \subseteq \mathbb{R}^{n+1}$ be as in Definition 2.9. Let $t \in \mathbb{R}^+$ and let $P \subseteq t\Delta_n$ be a convex polytope. Then $\mathrm{vol}_n(\pi_n(P)) \geq \#(P \cap \mathbb{Z}^{n+1}) - (t+1)^n + t^n$.*

Proof. Let $C_n \subseteq \mathbb{R}^n$ denote the centered unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ and set $Q = \pi_n(P)$. Then we have

$$\#P \cap \mathbb{Z}^{n+1} \leq \#Q \cap \mathbb{Z}^n = \sum_{x \in Q \cap \mathbb{Z}^n} \mathrm{vol}(C_n) = \mathrm{vol}((Q \cap \mathbb{Z}^n) + C_n) \leq \mathrm{vol}(Q + C_n), \tag{1}$$

so it suffices to estimate $\mathrm{vol}(Q + C_n)$. By Theorem 3.4, we have

$$\mathrm{vol}(Q + C_n) = \sum_{i=0}^{n} \binom{n}{i} V_n(Q^{\langle i \rangle}, C_n^{\langle n-i \rangle}). \tag{2}$$

Since $\frac{1}{t}Q \subseteq \pi_n(\Delta_n)$ and $\pi_n(\Delta_n) \subseteq \{(\frac{1}{2}, \dots, \frac{1}{2})\} + C_n$, by Theorem 3.6, we have

$$V_n(Q^{\langle i \rangle}, Q_n^{\langle n-i \rangle}) = t^i V_n(\frac{1}{t}Q, C_n) \leq t^i \,\mathrm{vol}(C_n) = t^i. \tag{3}$$

Applying the bound Equation (3) to Equation (2) for all $0 \le i \le n-1$, we obtain

$$\operatorname{vol}(Q + C_n) \le \sum_{i=0}^{n-1} \binom{n}{i} t^i + \operatorname{vol}(Q) = (t+1)^n - t^n + \operatorname{vol}(Q). \tag{4}$$

Combining Equations (1) and (4) and rearranging gives the result. □

3.3. **An Application of Grünbaum's Inequality.** We first recall a result of Grünbaum, which we state an equivalent version of below.

Theorem 3.9 ([5], Theorem 2)**.** *Let $K \subseteq \mathbb{R}^n$ be a convex body and let c denote the centroid of K. Let H^+ be a half-space whose boundary hyperplane contains c. Then*

$$\operatorname{vol}(H^+ \cap K) \le \left(1 - \left(\frac{n}{n+1}\right)^n\right) \operatorname{vol}(K).$$

Definition 3.10. We let $\mathcal{M}_n$ denote the quantity $\left(1 - \left(\frac{n}{n+1}\right)^n\right)$ from the theorem.

For our purposes, we must characterize the equality case of Theorem 3.9.

Proposition 3.11. *Suppose H^+, K are as in Theorem 3.9, with $\operatorname{vol}(K) > 0$ and $\operatorname{vol}(H^+ \cap K) = \mathcal{M}_n \operatorname{vol}(K)$. Let H denote the boundary hyperplane of H^+. Then there exists a convex body $K' \subseteq H^+ \cap K$ and a point $q \in H^- \cap K$ such that K' is contained in a hyperplane parallel to H and $K = \operatorname{conv}(K' \cup \{q\})$.*

Proof. Follows from [12], Corollary 8. □

Lemma 3.12. *Let $T_n := \pi_n(\Delta_n)$. Let $z_n = (\frac{1}{n+1}, \dots, \frac{1}{n+1})$ denote the centroid of T_n. Let H^+ be a half-space whose boundary hyperplane H contains z_n. Then*

$$\operatorname{vol}(H^+ \cap T_n) \le \frac{\mathcal{M}_n}{n!}$$

with equality if and only if H is parallel to a facet F of T_n with $F \subseteq H^+$.

Proof. If K' is an $n-1$-dimensional convex set and q a point not contained in the hyperplane supporting K' such that $\operatorname{conv}(K' \cup \{q\})$ is a polytope, then K' is a facet of $\operatorname{conv}(K' \cup \{q\})$. The result therefore follows from Proposition 3.11. □

Lemma 3.13. *Let $P \subseteq T_n$ be a closed convex set with $z_n \notin intP$. Then $\operatorname{vol}(P) \le \mathcal{M}_n/n!$ with equality if and only if P is the intersection of T_n with a half-space satisfying the conditions of Lemma 3.12.*

Proof. First, we note that it suffices to consider P with $z_n \in \partial P$. To see this, if $z_n \notin \partial P$, then for $0 < \varepsilon \le \operatorname{dist}_{L^1}(P, z_n)$, the set $(P + \varepsilon C_n) \cap T_n$ is a strictly larger convex set which does not contain z_n in its interior.

Moving forward, we assume $z_n \in \partial P$. Let $\chi_P : T_n \to [0,1]$ denote the characteristic function of P. Since $-\chi_P$ is proper and convex, there exists a subgradient v to $-\chi_P$ at z_n. Let H^- denote the set $\{x \in \mathbb{R}^n : \langle v, (x - z_n) \rangle \le 0\}$. As

$$\langle v, (x - z_n) \rangle \le -\chi_P(x) + \chi_P(z_n) = 1 - \chi_P(x)$$

for all $x \in T_n$, we have $P \subseteq H^-$. We have $\operatorname{vol}(P) \le \operatorname{vol}(T_n \cap H^-)$ with equality if and only if $P = T_n \cap H^-$. It suffices, therefore, to show prove the claim for $P = H^- \cap T_n$. This, however, is immediate from Lemma 3.12. □

3.4. **Proof of Theorem 3.17.**

Lemma 3.14. *Let k be an an algebraically-closed field of charactestic $p > 0$ and let $R = k[x_0, \dots, x_n]$. Let $I = (f_1, \dots, f_n) \subseteq R$ denote a complete intersection ideal generated by d-forms. Then* $\operatorname{fpt}(I) \geq n/d$, *with equality if and only if $\overline{I} = (x_1, \dots, x_n)^d$ up to change of coordinates.*

We begin with a computation of the Hilbert series of R/I^s.

Lemma 3.15. *Let I, R be as in Lemma 3.14. For $t \geq d(s-1) + (n-1)d - n + 2$, we have $H_R(R/I^s, t) = \binom{n+s-1}{n} d^r$. In particular, this holds for $t \geq d(s+n)$.*

Proof. We define

$$\mathcal{L}_{n,s} := \{(a_1, \dots, a_n) : a_i \geq 0, a_1 + \cdots + a_n \leq s-1\}.$$

By [6], Corollary 2.3, we have

$$H_R(R/I^s, t) = \sum_{(a_1,\dots,a_r)\in\mathcal{L}_{n,s}} H_R(R/I, t - d(a_1 + \cdots + a_n)).$$

The Koszul resolution of R/I shows that the CM-regularity of R/I is $d(n-1) - n + 1$, so $H_R(R/I, t)$ agrees with the Hilbert polynomial of R/I for $t \geq d(n-1) - n + 2$. The Hilbert polynomial of R/I is d^n, so for $t \geq d(s-1) + (n-1)d - n + 2$, we have

$$H_R(R/I^s, t) = |\mathcal{L}_{n,s}| d^n = \binom{n+s-1}{n} d^n.$$

The second part of the statement follows from the bound

$$d(s-1) + (n-1)d - n + 2 \leq (ds - 1) + (nd - 1) + 2 = d(s+n).$$

$\square$

Lemma 3.16. *Let $\mathfrak{a} \subseteq R$ be a monomial ideal containing a monomial m of degree t. For any $t' > t$, if $\frac{t'}{n+1}\vec{1} \in \gamma(\mathfrak{a}, t')$, then* $\operatorname{fpt}(\mathfrak{a}) > \frac{n+1}{t'}$.

Proof. Set $y = \log(m)$. By convexity of $\Gamma(\mathfrak{a})$, we have $\lambda y + (1-\lambda)\gamma(\mathfrak{a}, t') \subseteq \Gamma(\mathfrak{a})$ for all $\lambda \in [0,1]$. Taking $0 < \lambda \ll 1$, we obtain $\frac{\lambda t + (1-\lambda)t'}{n+1}\vec{1} \in \Gamma(\mathfrak{a})$, which implies $\operatorname{fpt}(\mathfrak{a}) \geq \frac{n+1}{\lambda t + (1-\lambda)t'} > \frac{n+1}{t'}$. $\square$

We now prove Lemma 3.14.

Proof. Let $\mathfrak{p}$ be a minimal prime over I. Since $k = \overline{k}$ and I is homogeneous, we may change coordinates so that $\mathfrak{p} = (x_1, \dots, x_n)$. Let $>$ denote the lexicographic order, and define the graded system of ideals $\mathfrak{a}_\bullet = \{\operatorname{in}_>(I^{nm})\}_m$. Since $\mathfrak{p}^r$ is a monomial ideal for all $r \geq 0$ and $I^r \subseteq \mathfrak{p}^r$, we have $\mathfrak{a}_m \subseteq \mathfrak{p}^{nm}$ for all $m \geq 0$. Since $\mathfrak{a}_\bullet$ is graded, we have for any $t \in \mathbb{Z}^+$

$$[\mathfrak{a}_{2^m}]_{2^m t}[\mathfrak{a}_{2^m}]_{2^m t} \subseteq [\mathfrak{a}_{2^m}\mathfrak{a}_{2^m}]_{2^{m+1}t} \subseteq [\mathfrak{a}_{2^{m+1}}]_{2^{m+1}t}.$$

It follows that $\{\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m t)\}_m$ is an ascending chain of convex subsets of H_t. We then set $t = d(n+1)$ and let $\mathcal{P}$ denote the ascending union $\bigcup_{m\geq 1}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. If $d\vec{1} \in \mathcal{P}$, there exists some m such that $d\vec{1} \in \gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. By Lemma 3.16, we have $\operatorname{fpt}(\mathfrak{a}_{2^m}) > \frac{n+1}{2^m d(n+1)} = \frac{1}{2^m d}$, so $\operatorname{fpt}(I) > n/d$.

Conversely, suppose $d\vec{1} \notin \mathcal{P}$. Then for all m, we have $d\vec{1} \notin \frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))$. By Lemma 3.13, we have

$$\operatorname{vol}(\mathcal{P}) = \lim_{m\to\infty} \operatorname{vol}\left(\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m}, 2^m d(n+1))\right) \leq (d(n+1))^n \frac{\mathcal{M}_n}{n!}. \tag{5}$$

We now derive a lower bound for $\mathrm{vol}(\mathcal{P})$. First, by Lemma 3.15, we have

$$\begin{aligned}\#\mathbb{Z}^{n+1}\cap(\gamma(\mathfrak{a}_{2^m},2^md(n+1))) &\geq H_R(\mathfrak{a}_{2^m},2^md(n+1))\\ &= H_R(I^{2^mn},2^md(n+1))\\ &= \binom{n+2^md(n+1)}{n}-\binom{n+2^mn-1}{n}d^n\end{aligned}$$

provided $2^md(n+1)\geq d(2^mn+n)$, which is satisfied for all $m\geq\log_2 n$. Using the approximation $\binom{a+b}{b}=\frac{a^b}{b!}+O_b(a^{b-1})$, we have

$$\#\mathbb{Z}^{n+1}\cap(\gamma(\mathfrak{a}_{2^m},2^md(n+1)))\geq\frac{(2^md)^n}{n!}\left((n+1)^n-n^n\right)+O(2^{m(n-1)}) \tag{6}$$

Consequently, by Lemma 3.8 we have

$$\begin{aligned}\mathrm{vol}(\mathcal{P}) &= \lim_{m\to\infty}\mathrm{vol}\left(\frac{1}{2^m}\gamma(\mathfrak{a}_{2^m},2^md(n+1))\right)=\lim_{m\to\infty}\frac{1}{2^{mn}}\mathrm{vol}(\gamma(\mathfrak{a}_{2^m},2^md(n+1)))\\ &\geq \lim_{m\to\infty}\frac{1}{2^{mn}}\frac{(2^md)^n}{n!}\left((n+1)^n-n^n\right)+O(2^{m(n-1)})\\ &= (d(n+1))^n\frac{\mathcal{M}_n}{n!}.\end{aligned}$$

It follows that $\mathrm{vol}(\mathcal{P})=\mathrm{vol}(\overline{\mathcal{P}})=(d(n+1))^n\frac{\mathcal{M}_n}{n!}$, so by Lemma 3.12, we have $\overline{\mathcal{P}}=H^+\cap(d(n+1))\Delta_n$. Moreover, the boundary hyperplane H of H^+ is parallel to a facet F of $(d(n+1))\Delta_n$ with $F\subseteq H^+$ and $d(n+1)\eta_n\in H$.

For $\alpha\in\mathbb{R}$, define $D_{t,\beta}=\{(a_0,\dots,a_n)\in t\Delta_n : y_0\leq\beta\}$. Since $\mathfrak{a}_m\subseteq\mathfrak{p}^{mn}$, for any monomial $x_0^{a_0}\dots x_n^{a_n}\in(\mathfrak{a}_m)_t$, we have $a_1+\dots+a_n\geq mn$ and hence $a_0\leq t-mn$. In particular, for all $m\geq 0$ we have

$$\gamma(\mathfrak{a}_{2^m},2^md(n+1))\subseteq D_{2^md(n+1),2^mn(d+1)-2^mn}.$$

As a consequence, we conclude $\mathcal{P}\subseteq D_{d(n+1),d(n+1)-n}$. It follows that the facet F is the facet $\{a_0=0\}$, so we conclude $\overline{\mathcal{P}}=D_{d(n+1),d}$. We then have

$$\Gamma(\mathfrak{a}_1,d(n+1))\subseteq\overline{\mathcal{P}}=D_{d(n+1),d}=\Gamma(\mathfrak{p}^{nd},d(n+1)),$$

so $[\mathfrak{a}_1]_{d(n+1)}\subseteq[\mathfrak{p}^{nd}]_{n(d+1)}$. For each generator f_i of I, we have $x_0^d\,\mathrm{in}_>(f_i^n)\in[\mathfrak{a}_1]_{d(n+1)}\subseteq[\mathfrak{p}^{nd}]_{n(d+1)}$, so $x_0\nmid\mathrm{in}_>(f_i^n)$ for all i. As $\mathrm{in}_>(f_i^n)=\mathrm{in}_>(f_i)^n$, we deduce that $I\subseteq\mathfrak{p}^d$.

As the generators of I are d-forms contained in $(x_1,\dots,x_n)^d$, we have that I is extended from an ideal $I'\subseteq k[x_1,\dots,x_n]$. It follows from Theorem 2.1 that $\overline{I}=\overline{I'}R=\mathfrak{p}^d$. □

Theorem 3.17. *Let k be an algebraically-closed field of characteristic $p>0$. Let I be a homogeneous ideal in $k[x_0,\dots,x_n]$ generated by d-forms and set $h=\mathrm{codim}(I)$. Then $\mathrm{fpt}(I)=h/d$ if and only if $\overline{I}=(x_0,\dots,x_{h-1}2)^d$ up to change of coordinates.*

Proof. Let k be an algebraically-closed field and $R=k[0_1,\dots,x_n]$. Let $I\subseteq R$ be an ideal generated by d-forms, and suppose that $\mathrm{codim}(I)=n$. If $f_1,\dots,f_n$ are $n-1$ general d-forms in I, then $J=(f_1,\dots,f_n)$ is a complete intersection. By Lemma 3.14, we may change coordinates on R such that $\overline{J}=(x_1,\dots,x_n)^d$. Then we have $(x_1,\dots,x_n)^d\subseteq\overline{I}$. Let $>$ denote the lexicographic order, and let g be a d-form in $\overline{I}$. Write $\mathrm{in}_>(g)=x_0^{a_0}\cdots x_n^{a_n}$. Set $a=\max_i a_i$. Then

$$g^{\lfloor(p^e-1)/a\rfloor}\prod_{i=1}^n(x_i^d)^{\lfloor((p^e-1)-a_i\lfloor(p^e-1)/a\rfloor)/d\rfloor}\notin\mathfrak{m}^{[p^e]},$$

so we have

$$\mathrm{fpt}(\mathrm{in}_>(\overline{I}))\geq\frac{1}{a}+\sum_{i=1}^n\left(\frac{1}{d}-\frac{a_i}{ad}\right)=\frac{n}{d}+\frac{a_0}{ad}.$$

Consequently, we have

$$\frac{n}{d} = \operatorname{fpt}(I) = \operatorname{fpt}(\overline{I}) \geq \operatorname{fpt}(\operatorname{in}_>(\overline{I})) \geq \operatorname{fpt}((x_1,\dots,x_n)^d + (x_0^{a_1}\cdots x_n^{a_n})) = \frac{n}{d} + \frac{a_0}{ad},$$

so we have $a_0 = 0$, hence $\operatorname{in}_>(g) \in (x_1,\dots,x_n)^d$. As $>$ is the lexicographic order, it follows that $g \in (x_1,\dots,x_n)^d$. As g was arbitrary, we conclude that $\overline{I} = (x_1,\dots,x_n)^d$.

Next, we consider the case that $\operatorname{codim} I \neq n$. If $\operatorname{codim} I = n+1$, then $\overline{I} = (x_0,\dots,x_n)^{n+1}$ by Theorem 2.1. Otherwise, suppose $\operatorname{codim} I = h \leq n-1$. Let L be an ideal generated by $n-h$ linear forms. Then $\frac{h}{d} \leq \operatorname{fpt}(\frac{I+L}{L}) \leq \frac{h}{d}$, so by the case where R/I is one-dimensional, we have that $\overline{\frac{I+L}{L}}$ is equal to $\frac{(x_1,\dots,x_h)^d+L}{L}$ up to a change of variables. By Proposition 3.3, the same holds for I. □

4. A New Bertini Theorem for F-Purity of Pairs

Lemma 4.1 ([1], Lemma 3.2)**.** *Let $R := k[x_0,\dots,x_n], \mathfrak{m} := (x_0,\dots,x_n)$. For $e,t \in \mathbb{Z}^+$, we have*

$$(\mathfrak{m}^{[p^e]} : \mathfrak{m}^t) = \begin{cases} R & t \geq (n+1)p^e - n \\ \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n+1)p^e-n-t} & t < np^e - n + 1 \end{cases}$$

Lemma 4.2. *Let k be a field of characteristic $p > 0$, let $R = k[x_1,\dots,x_n]$, and $I \subseteq \mathfrak{m}$ a homogeneous ideal. For $H = V(\ell) \in (\mathbb{P}^n)^\vee$, we let $I|_H$ denote the image of I in R/ℓ. In this case, we have*

$$\nu_{I|_H}(p^e) \leq \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{n(p^e-1)+1}\}, \tag{7}$$

Conversely, if $|k| \geq p^e$, then there exists $H \in (\mathbb{P}^n)^\vee(k)$ such that

$$\nu_{I|_H}(p^e) \geq \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{n(p^e-1)-(n-1)(p^{e-1})+1}\} \tag{8}$$

Proof. Let $\mathfrak{a}_e := \mathfrak{m}^{[p^e]} + \mathfrak{m}^{n(p^e-1)+1}, d_e = n(p^e-1)-(n-1)(p^{e-1})+1$, and $\mathfrak{b}_e := \mathfrak{m}^{[p^e]} + \mathfrak{m}^{d_e}$. We have $\mathfrak{a}|_H = \mathfrak{m}^{[p^e]}|_H$, which proves the bound 7. Conversely, suppose $f \in I^r \setminus \mathfrak{b}_e$ is a homogeneous element. We may assume $\deg f = d_e$. Write

$$f = \sum_{a_0+\cdots+a_n=d_e} c_{a_0,\dots,a_n} x_0^{a_0}\cdots x_n^{a_n}.$$

For $\lambda \in k^n$, let H_λ denote the hyperplane cut out by $x_0 = \lambda_1 x_1 + \cdots + \lambda_n x_n$. For $b_1,\dots,b_n \in \mathbb{Z}^{\geq 0}$ such that $b_1 + \cdots + b_n = d_e$, define

$$P_{b_1,\dots,b_n}(\lambda) := \sum_{a_0=0}^{d_e} \left(\sum_{\substack{a_i \leq b_i \ \forall 1\leq i\leq n \\ a_1+\cdots+a_n = d_e - a_0}} c_{a_0,\dots,a_n} \binom{a_0}{b_1-a_1,\dots,b_n-a_n} \lambda_1^{b_1-a_1}\cdots\lambda_n^{b_n-a_n} \right).$$

Then we have

$$\begin{aligned} f|_H &= \sum_{a_0+\cdots+a_n=d_e} c_{a_0,\dots,a_n} x_1^{a_1}\cdots x_n^{a_n}(\lambda_1 x_1 + \dots \lambda_n x_n)^{a_0} \\ &= \sum_{b_1+\cdots+b_n=d_e} x_1^{b_1}\dots x_n^{b_n} P_{b_1,\dots,b_n}(\lambda). \end{aligned}$$

To prove that $f|_H \notin \mathfrak{m}^{[p^e]}$ for some H, we will first prove that there exist $b_1,\dots,b_n$ for which $P_{b_1,\dots,b_n}(\lambda)$ is a nonzero polynomial in λ. To this end, it suffices to produce $a_0,\dots,a_n,b_1,\dots,b_n$ such that

(i) $a_0 + \cdots + a_n = d_e$
(ii) $c_{a_0,\dots,a_n} \neq 0$
(iii) $b_1 + \cdots + b_n = d_e$
(iv) $a_i \leq b_i \leq p^e - 1$ for all $1 \leq i \leq n$

(v) We have
$$\binom{a_0}{b_1-a_1,\dots,b_n-a_n} \not\equiv 0 \mod p.$$

By assumption that $f \notin \mathfrak{m}^{[p^e]}$, it is possible to choose $a_0,\dots,a_n$ such that $a_0+\cdots+a_n = d_e, a_0,\dots,a_n \le p^e-1$, and $c_{a_0,\dots,a_n} \neq 0$. We will prove, by induction on the p-ary digits of a_0, that there exist $b_1,\dots,b_n$ satisfying (iii)-(v). The base case, when $a_0=0$, is obvious. Write $a_0 = \alpha_0+\alpha_1 p+\cdots+\alpha_{e-1}p^{e-1}$ and suppose $\alpha_j \neq 0$.

By the pigeonhole principle, we have
$$\max_{1\le i\le n}(p^e-1)-a_i \ge p^e - \frac{d_e-a_0}{n} \ge p^e - \frac{d_e-p^j}{n} = \frac{(n-1)(p^{e-1})}{n} + \frac{p^j}{n} \ge p^j.$$

It follows that there exists some $1\le i\le n$ with $a_i+p^j \le p^e-1$. We apply the induction hypothesis to produce integers $b_1,\dots,b_n$ satisfying (iii)-(v) with respect to $(a_0-p^j, a_1,\dots,a_i+p^j,\dots,a_n)$. Since
$$\binom{a_0-p_j}{b_1-a_1,\dots,b_i-a_i-p_j,\dots,b_n-a_n} \not\equiv 0 \mod p,$$
it follows by Lucas's theorem that we can perform the addition
$$(b_1-a_1)+\cdots+(b_i-a_i-p^j)+\cdots+(b_n-a_n) = a_0-p^j$$
in base p without having to carry a digit. Consequently, the same is true for the addition
$$(b_1-a_1)+\cdots+(b_i-a_i)+\cdots+(b_n-a_n) = a_0,$$
so $b_1,\dots,b_n$ satisfy conditions (iii)-(v) for the original tuple $(a_1,\dots,a_n)$.

By the above analysis, the coefficient of $x_1^{b_1}\cdots x_{n-1}^{b_n}$ in $f|_H$ is a nonzero polynomial of total degree $a_n \le p^e-1$ in the variables $\lambda_1,\dots,\lambda_n$. By the Schwarz-Zippel lemma [SRC], there exist $\lambda_1,\dots,\lambda_{n-1}$ for which the coefficient is nonzero, proving the claim. $\square$

As a consequence, we have the following.

Theorem 4.3. *Let k be an infinite field of characteristic $p>0$. Let $R=k[x_0,\dots,x_n]$. Let $I\subseteq R$ be an ideal generated by forms of degree at most d. Let $H\in(\mathbb{P}^n)^\vee$ be a general hyperplane through the origin. Then for all $0\le t<\frac{n}{d}-\frac{n-1}{pd}$, the pair (R,I^t) is sharply F-split if and only if $(H, I^t|_H)$ is sharply F-split.*

Proof. The implication $(H,I^t|_H)$ sharply F-split $\implies$ (R,I^t) sharply F-split is well-known and additionally is immediate from Lemma 4.2. Conversely, suppose (R,I^t) is sharply F-split and $t<\frac{n}{d}-\frac{n-1}{pd}$. Then there exists $M\in\mathbb{Z}^+$ such that for all $e\in\mathbb{Z}^+, M\mid e$, we have $I^{\lceil t(p^e-1)\rceil}\not\subseteq \mathfrak{m}^{[p^e]}$. Since $t<\frac{n}{d}-\frac{n-1}{pd}$, we may choose $e\gg 0$ divisible by M such that $td<n(p^e-1)-(n-1)(p^{e-1})+1$, whence we have $I|_H^{\lceil t(p^e-1)\rceil}\not\subseteq \mathfrak{m}^{[p^e]}|_H$ by Lemma 4.2. $\square$

In terms of the F-pure threshold, Theorem 4.3 says the following.

Corollary 4.4. *Let R,I,H be as in Theorem 4.3. Then*
$$\min\left(\frac{n}{d}-\frac{n-1}{pd}, \operatorname{fpt}(I)\right) \le \operatorname{fpt}(I|_H) \le \min\left(\frac{n}{d}, \operatorname{fpt}(I)\right).$$

Proof. For the first inequality, we note by Theorem 4.3 that $(R|_H, I^t|_H)$ is sharply F-split for all $\min\left(\frac{n}{d}-\frac{n-1}{pd}, \operatorname{fpt}(I)\right)$. For the second inequality, we have by Lemma 4.2
$$\begin{aligned}\operatorname{fpt}(I|_H) &\le \lim_{e\to\infty} p^{-e}\sup\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]}+\mathfrak{m}^{n(p^e-1)+1}\}\\ &\le \lim_{e\to\infty} p^{-e}\min\left(\nu_I(p^e), \left\lfloor\frac{n(p^e-1)+1}{d}\right\rfloor\right) = \min\left(\operatorname{fpt}(I), \frac{n}{d}\right).\end{aligned}$$

□

Example 4.5. In Theorem 4.3, our bound on t is optimal. If char $k = p$ and $R = k[x_0, \dots, x_n]$, then we may take $f = x_0(x_1 \cdots x_n)^{p-1}$ and $t = \frac{n}{n(p-1)+1} - \frac{n-1}{p(n(p-1)+1)} = \frac{1}{p}$. Then $\operatorname{fpt}(R, f) = \frac{1}{p-1}$, so (R, f^t) is sharply F-split. For any hyperplane $H \subseteq R$ we have $f|_H \in \mathfrak{m}|_H^{[p]}$, so $\operatorname{fpt} f|_H \le t$. Since p divides the denominator of t, we have that $(H, f^t|_H)$ is not sharply F-split.

5. The Test Ideal at the Threshold

Example 5.1. Let $R = \mathbb{F}_p[x, y, z]$ and $f = (x^3 + y^3 + z^3)$. If $p \equiv 2 \mod 3$, then $\operatorname{fpt}(f) = 1 - 1/p$. The non-strongly F-regular locus of $(R, f^{1-1/p})$ is cut out by $\tau(R, f^{1-1/p})$. Since the coefficient of $x^{p+1}y^{p-2}z^{p-2}$ is nonzero in f^{p-1}, one can verify that $x \in \tau(R, f^{1-1/p})$, and so $\tau(R, f^{1-1/p}) = (x, y, z)$ by symmetry. But then $e = d = 3$, and $1 - 1/p < 1$.

If we impose the additional condition that the pair is F-split at the threshold, then there is no longer any issue.

Lemma 5.2. *Let k be a field of characteristic $p > 0$. Let $R = k[x_0, \dots, x_n]$. Suppose $I \subseteq R$ is generated by homogeneous polynomials of degree d. Suppose $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split and let $h = \operatorname{codim}(\tau(R, I^{\operatorname{fpt}(I)}))$. Suppose further that $h \ge n$. Then $\operatorname{fpt}(I) \ge h/d$.*

Proof. Define the graded system of ideals $\mathfrak{a}_\bullet$ by $\mathfrak{a}_m = I^{\lfloor m \operatorname{fpt}(I) \rfloor}$. Let $\mathfrak{p}$ be a minimal prime over $\tau := \tau(R, I^{\operatorname{fpt}(I)})$. As $\mathfrak{p}$ is a homogeneous prime ideal of codimension n or $n+1$, we may change coordinates so that $\mathfrak{p} = (x_0, \dots, x_{h-1})$. By [13], Proposition 4.5 and 4.7, we have that $\mathfrak{p}$ is uniformly $(\mathfrak{a}_\bullet, F)$-compatible, so for all $e \ge 0$ we have $\mathfrak{a}_{p^e-1} \subseteq (\mathfrak{p}^{[p^e]} : \mathfrak{p}) = \mathfrak{p}^{[p^e]} + (x_0 \cdots x_{h-1})^{p^e-1}$. By assumption that $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split, there exists $M > 0$ such that for all $e \ge 0, M \mid e$ we have $\mathfrak{a}_{p^e-1} \not\subseteq \mathfrak{m}^{[p^e]}$. Let $M \mid e$, and let f be a generator of $\mathfrak{a}_{p^e-1}$ such that $f \in \mathfrak{p}^{[p^e]} + (x_0 \cdots x_{h-1})^{p^e-1} \setminus \mathfrak{m}^{[p^e]}$. Then $\lceil \operatorname{fpt}(I) d(p^e - 1) \rceil \deg f \ge h(p^e - 1)$, so $\operatorname{fpt}(I) \ge \frac{h}{d}$. □

In particular, by [7, Theorem 4.1], the hypothesis that $(R, I^{\operatorname{fpt}(I)})$ is sharply F-split is satisfied whenever I is principal and p does not divide the denominator of $\operatorname{fpt}(I)$.

Theorem 5.3. *Suppose* char $k = p > 0$ *and $R = k[x_0, \dots, x_n]$. Suppose that $f \in R_d$ is smooth in codimension c and p does not divide the denominator of* $\operatorname{fpt}(f)$. *Further suppose that $c \ge n$ or $p \ge c$. Then* $\operatorname{fpt}(f) \ge \min(c/d, 1)$.

Proof. Assume $\operatorname{fpt}(f) < 1$. We will first demonstrate that $V(\tau(R, f^{\operatorname{fpt}(f)})) \subseteq \operatorname{Sing}(R/f)$. To see this, suppose $\mathfrak{p}$ is a nonsingular point of R/f. Then in particular, $R_\mathfrak{p}/fR_\mathfrak{p}$ is F-split, so $\operatorname{fpt}(R_\mathfrak{p}, fR_\mathfrak{p}) = 1$ and $\tau(R_\mathfrak{p}, f^{\operatorname{fpt}(f)}R_\mathfrak{p}) = R_\mathfrak{p}$. It follows that $\mathfrak{p} \notin V(\tau(R, f^{\operatorname{fpt}(f)}))$. For $c \in \{n, n+1\}$, it follows that $\operatorname{fpt}(f) \ge \operatorname{codim}(\tau(R, f^{\operatorname{fpt}(f)}))/d \ge c/d$.

Suppose instead $c \le n - 1$ and $p \ge c$; we'll prove the claim by induction on $n + 1 - c$. Suppose for the sake of contradiction $\operatorname{fpt}(f) < c/d$. Let H be a general element of $(\mathbb{P}^n)^\vee$. Then $\operatorname{codim}(H, \operatorname{Sing}(f|_H)) = \operatorname{codim}(\operatorname{Spec} R, \operatorname{Sing}(f)) = c$ by Bertini's theorem. As

$$\operatorname{fpt}(f) < \frac{c}{d} \le \frac{c+1}{d} - \frac{c}{pd} \le \frac{n}{d} - \frac{c-1}{pd},$$

we have by Corollary 4.4 that $\operatorname{fpt}(f) = \operatorname{fpt}(f|_H)$. In particular, p does not divide the denominator of $\operatorname{fpt}(f)$. By induction, we conclude that $\operatorname{fpt}(f) < c/d \le \operatorname{fpt}(f|_H) = \operatorname{fpt}(f)$, a contradiction. □

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