CLASSIFICATION OF MINIMAL SINGULARITY THRESHOLDS

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ABSTRACT. Let k be a field of characteristic zero, $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ an ideal primary to $\mathfrak{m} = (x_1, \dots, x_n)$. By a 2014 result of Demailly and Pham, we have $\mathrm{lct}(I) \geq \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)}$, where $e_j(I)$ is the mixed multiplicity $e(I, \dots, I, \mathfrak{m}, \dots, \mathfrak{m})$, with I repeated j times and $\mathfrak{m}$ repeated $n-j$ times. If instead char $k = p > 0$, we show that the F-pure threshold of (R, I) satisfies the same lower bound. In both characteristic zero and positive characteristic, we classify all homogeneous ideals which attain the lower bound.

1. INTRODUCTION

We consider the log canonical threshold (lct) and F-pure threshold (fpt) of a pair (X, Y) where X is a smooth k-scheme and Y a subscheme supported at a point. The lct in characteristic zero and the fpt in positive characteristic have attracted considerable attention in algebraic geometry due to their connections with the Minimal Model Program and singularity theory. In recent years, many authors [4, 7, 9, 10, 11, 19] have proven results comparing the lct to multiplicity-like invariants of the pair (X, Y).

In this paper, we consider a lower bound on the lct due to Demailly and Pham [7] in terms of mixed multiplicities (Theorem 1.1). We show that in positive characteristic, the analogous bound holds for fpt (Corollary 3.12). Our main contribution (Theorem 4.22) is to classify the homogeneous pairs (X, Y) for which the lct or fpt equals the lower bound.

Theorem 1.1 ([7], Theorem 1.2). *Let $(\mathcal{O}_n, \mathfrak{m})$ denote the ring of germs at zero of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$. Let I be an $\mathfrak{m}$-primary ideal and let $e_j(I)$ denote the mixed multiplicity $e(I, \dots, I, \mathfrak{m}, \dots, \mathfrak{m})$ where I is repeated j times and $\mathfrak{m}$ repeated $n-j$ times (see Section 2.2). Then we have*

$$\frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} \leq \mathrm{lct}(I). \tag{1}$$

Moreover, this bound is attained by the ideal $I = (x_1^{d_1}, \dots, x_n^{d_n})$ for any $d_1, \dots, d_n \in \mathbb{Z}^+$.

We refer to the left-hand side of Equation (1) as the Demailly-Pham invariant of I, denoted $DP(I)$ (see Section 2.2). In this paper, we classify homogeneous ideals I that achieve equality in Equation (1).

Theorem 4.22. *Let k be an algebraically-closed field of characteristic zero. Let $R = k[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and let $I \subseteq R$ be a $\mathfrak{m}$-primary homogeneous ideal. If $DP(I) = \mathrm{lct}(I)$, then there exist integers $d_1, \dots, d_n$ such that, in suitable coordinates, we have*

$$\overline{I} = \overline{\left(x_1^{d_1}, \dots, x_n^{d_n}\right)}.$$

If instead char $k = p > 0$*, then the same result holds with* $\mathrm{lct}(I)$ *replaced by* $\mathrm{fpt}(I)$.

We briefly outline the proof of the theorem. Assume I is an ideal satisfying equality in Equation (1). Write $I = I_1 + \cdots + I_r$, where I_j is generated by d_j-forms.

(1) Using results from [4], we control the generic initial ideals $\{\mathrm{gin}(I^n)\}_{n \geq 1}$.

The author was supported by NSF grant DMS-2101075 and NSF RTG grant DMS-1840234.

(2) Using (1), we obtain a formula for $e_j(I)$ in terms of the numbers $d_j, \operatorname{codim}(I_1 + \cdots + I_j)$, which allows us to reduce to the case of a complete intersection.
(3) We prove the result by induction on the number of distinct degrees $d_1, \ldots, d_r$.

In the case $r = 1$, any $\mathfrak{m}$-primary ideal I generated by d-forms automatically satisfies $\overline{I} = \mathfrak{m}^d$, so there is no way to use $r = 1$ as a useful base case for our induction. Instead, we use $r = 2$. In this case, we show (Lemma 3.19) that $c(I) = DP(I)$ if and only if $c(I_1) = \operatorname{codim}(I_1)/d_1$. As $\overline{I} = \overline{\overline{I_1} + \mathfrak{m}^{d_2}}$, it suffices to show that $\overline{I_1} = (x_1, \ldots, x_{\operatorname{codim}(I)})^{d_1}$ in suitable coordinates. In characteristic zero, this follows from [10, Theorem 3.5]. In positive characteristic, this fact is recorded below.

Theorem 1.2 ([1], Theorem 3.17). *Let k be a field of characteristic $p > 0$. Let I be a homogeneous ideal in $k[x_1, \ldots, x_n]$ generated by polynomials of degree d and set $h = \operatorname{codim}(I)$. Suppose that k is algebraically-closed. Then $\operatorname{fpt}(I) = h/d$ if and only if $\overline{I} = (x_1, \ldots, x_h)^d$ up to change of coordinates.*

2. Preliminaries

2.1. F-Pure and Log Canonical Thresholds.

We begin with a formal definition of the lct. For a detailed introduction, see [20].

Definition 2.1 (Log Resolution). Let X be a smooth variety over a characteristic zero field with $Y \subseteq X$ a proper closed subvariety with defining ideal $\mathfrak{a}$. Let W be a smooth variety. A projective morphism $\pi : W \to X$ is a *log resolution* of (X, Y) if π is an isomorphism over $X \setminus Y$ and the inverse image $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a Cartier divisor D such that $D + K_{W/X}$ has simple normal crossings.

The following result gives a concise definition of the lct.

Definition 2.2 (Log canonical threshold, [20] Theorem 1.1). Let X be a smooth variety with $Y \subseteq X$ a closed subvariety with defining ideal $\mathfrak{a}$. By Hironaka's theorem on resolution of singularities in characteristic zero, there exists a log resolution $\pi : W \to X$ of the pair (X, Y). If $E_1, \ldots, E_N$ are the exceptional divisors of π, then we can write

$$D = \sum_{i=1}^{N} a_i E_i \qquad \text{and} \qquad K_{W/X} = \sum_{i=1}^{N} k_i E_i.$$

The quantity $\min_i \frac{k_i+1}{a_i}$ does not depend on π and is called the **log canonical threshold** of (X, Y).

For detailed background on the F-pure threshold, we direct the reader to [24, 25]. In this subsection, we summarize several key definitions and results.

Definition 2.3. Let R be a ring of characteristic $p > 0$. We let F_*R denote the R-module structure on R given by restriction of scalars along the Frobenius map $F : R \to R$. We say R is F-finite if F_*R is module-finite over R.

Definition 2.4 ([24]). Let R be an F-finite ring, $I \subseteq R$ an ideal, and $t \in \mathbb{R}^+$. The pair (R, I^t) is *sharply F-split* if for some (equivalently, infinitely many) $e > 0$, the map

$$I^{\lceil t(p^e-1) \rceil} \cdot \operatorname{Hom}(F^e_* R, R) \to R$$

is surjective.

Definition 2.5 ([25]). The *F-pure threshold* of the pair (R, I) is the supremum of all t such that (R, I^t) is sharply F-split. We denote this quantity by $\operatorname{fpt}(R, I)$, or $\operatorname{fpt}(I)$ when the ambient ring is clear.

In practice, we do not use the above definitions. Instead, we use the following two propositions characterizing lct and fpt respectively.

Proposition 2.6. *Let $(R,\mathfrak{m})$ be an F-finite regular local ring. Then the F-pure threshold of the pair (R, I^t) is equal to*

$$\sup\left\{\frac{\nu}{p^e} : I^\nu \notin \mathfrak{m}^{[p^e]}\right\}.$$

In fact, let $\nu_I(p^e) = \max\{r : I^r \notin \mathfrak{m}^{[p^e]}\}$. Then the F-pure threshold of $(R,\mathfrak{a})$ is equal to the limit $\lim_{e\to\infty} \nu_I(p^e)/p^e$. If instead R is a polynomial ring over an F-finite field and $I \subseteq R$ a homogeneous ideal, then the same results hold when we let $\mathfrak{m}$ denote the homogeneous maximal ideal of R.

Proof. The first claim follows from [25, Lemma 3.9]. The existence of the limit is [22, Lemma 1.1]. For the graded setting, see [5, Proposition 3.10]. □

Proposition 2.7 ([13], Theorem 6.8)**.** *Let A be a finite-type $\mathbb{Z}$-algebra and $\mathfrak{a} \subseteq A[x_1,\ldots,x_n]$ an ideal. Set $k = Frac(A)$. Then we have*

$$\operatorname{lct}(k[x_1,\ldots,x_n], \mathfrak{a}\otimes_A k) = \lim_{\mu\in\max\operatorname{Spec} A, |A/\mu|\to\infty} \operatorname{fpt}(A/\mu[x_1,\ldots,x_n], \mathfrak{a}\otimes_A A/\mu).$$

Many of our results make sense for both fpt and lct, so we introduce the following notation to avoid stating the same results once each for characteristic zero and positive characteristic.

Notation 2.8. *Let $R = k[x_1,\ldots,x_n]$ and $I \subseteq R$ a homogeneous ideal. We define the quantity $c(R,I)$ as follows:*

$$c(R,I) = \begin{cases} \operatorname{fpt}(R,I) & \operatorname{char} R = p > 0 \\ \operatorname{lct}(R,I) & \operatorname{char} R = 0 \end{cases}.$$

If the ambient ring is clear, we will use $c(I)$ for short.

We will require the following essential facts.

Proposition 2.9 (Properties of the singularity threshold)**.** *Let $R = k[x_1,\ldots,x_n]$. Then for all ideals $I \subseteq R$ such that I contains a nonzerodivisor, we have*

(i) If $I \subseteq J$, then $c(I) \leq c(J)$.
(ii) For all $m > 0$, we have $c(I^m) = m^{-1}c(I)$.
(iii) We have $c(I) = c(\overline{I})$, where $\overline{I}$ denotes the integral closure of I.

Proof. For characteristic zero, see [20, Properties 1.12, 1.13, 1.15]. For characteristic $p > 0$, see [25, Proposition 2.2 (1), (2), (6).] □

Proposition 2.10. *Let $R = k[x_1,\ldots,x_n]$. Let $>$ be a monomial order. Let $I \subseteq R$ be an ideal, and $\operatorname{in}_>(I)$ the initial ideal of I with respect to $>$. Then $c(\operatorname{in}_>(I)) \leq c(I)$.*

Proof. For characteristic zero, see [6] for the semicontinuity of the lc threshold. For positive characteristic, see [25], the claim preceding Remark 4.6. □

2.2. **Mixed Multiplicities and the Demailly-Pham Invariant.**

To begin, we recall the definition of the mixed multiplicity symbol $e(I_1,\ldots,I_d;M)$.

Definition 2.11. Let M be a finite-length R-module. We let $\lambda_R(M)$ denote the length of M as an R-module.

Theorem 2.12 ([17], Theorem 17.4.2)**.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $I_1,\ldots,I_k$ ideals of R primary to $\mathfrak{m}$, and M a finitely-generated R-module. Then there exists a polynomial $P(n_1,\ldots,n_k)$ with rational coefficients and total degree at most $\dim R$ such that for all $n_1,\ldots,n_k \gg 0$, we have*

$$P(n_1,\ldots,n_k) = \lambda_R\left(\frac{M}{I_1^{n_1}\ldots I_k^{n_k}M}\right).$$

Remark 2.13. Suppose instead that S is a Noetherian ring, not necessarily local, and $\mathfrak{n}$ is any maximal ideal of S. If $I_1, \dots, I_k$ are $\mathfrak{n}$-primary ideals in S, then $I_1^{n_1} \cdots I_k^{n_k}$ is $\mathfrak{n}$-primary for all $n_1, \dots, n_k > 0$. Consequently, we have

$$\lambda_S \left(\frac{S}{I_1^{n_1} \cdots I_k^{n_k} S} \right) = \lambda_{S_\mathfrak{n}} \left(\frac{S_\mathfrak{n}}{I_1^{n_1} \cdots I_k^{n_k} S_\mathfrak{n}} \right)$$

for all $n_1, \dots, n_k$, so Theorem 2.12 holds for $I_1, \dots, I_k$ without assuming that S is local.

Definition 2.14 (Mixed Multiplicity)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $I_1, \dots, I_k$ be $\mathfrak{m}$-primary ideals of R. Let $Q(n_1, \dots, n_k)$ denote the degree-d part of $P(n_1, \dots, n_k)$. The coefficients of Q define the mixed multiplicities $e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M)$:

$$Q(n_1, \dots, n_k) = \sum_{d_1 + \cdots + d_k = d} \binom{d}{d_1, \dots, d_k}^{-1} e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M) \tag{2}$$

The expression $e(I_1^{\langle d_1 \rangle}, \dots, I_k^{\langle d_k \rangle}; M)$ is shorthand for the expression $e(I_1, \dots, I_1, \dots, I_k, \dots, I_d; M)$, where I_j is repeated d_j times.

Remark 2.15. Other authors, such as [17], have used the notation $e(I_1^{[d_1]}, \dots, I_k^{[d_k]}; M)$ instead. To avoid confusion with the Frobenius powers of the ideals I_j, we use angle brackets in the exponent.

We now define the mixed multiplicities $e_j(I)$.

Definition 2.16. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I denote an $\mathfrak{m}$-primary ideal. We define

$$e_j(I) = e(I^{\langle j \rangle}, \mathfrak{m}^{\langle d-j \rangle}; R).$$

Suppose instead $R = k[x_1, \dots, x_n]$ is a polynomial ring over a field. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let I be an $\mathfrak{m}$-primary ideal. By Remark 2.13, the function $\lambda_R(R/I^{n_1}\mathfrak{m}^{n_2}R)$ is a polynomial for $n_1, n_2 \gg 0$. We may therefore define $e_j(I)$ in terms of this polynomial, and this definition agrees with the quantity $e_j(IR_\mathfrak{m})$.

We record a few basic properties of the numbers $e_j(I)$ in a polynomial ring.

Proposition 2.17. *Let $R = k[x_1, \dots, x_n]$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let I be an $\mathfrak{m}$-primary ideal.*

(i) We have $e_0(I) = 1, e_1(I) = ord_\mathfrak{m}(I)$, and $e_n(I) = e(I)$.
(ii) The sequence $e_0(I), \dots, e_n(I)$ is log convex.
(iii) If $h_1, \dots, h_n$ are general 1-forms, then for all $0 \le j \le n$ we have $e_j(I) = e\left(\frac{I + (h_1, \dots, h_{n-j})}{(h_1, \dots, h_{n-j})}\right)$, where $e(-)$ denotes the usual Hilbert multiplicity.

Proof.
(i): Follows from (iii).
(ii): See [17], Theorem 17.7.2.
(iii): See [17], Corollary 17.4.7.

□

We will now define the Demailly-Pham invariant, first defined in [7] and named in [4].

Definition 2.18. Let $R = k[x_1, \dots, x_n]$, $\mathfrak{m}$ the homogeneous maximal ideal of R, and I an $\mathfrak{m}$-primary ideal. Then we set

$$DP(I) := \frac{1}{e_1(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)}.$$

This invariant satisfies a property similar to Theorem 2.37.

Proposition 2.19. *Assume the setting of Definition 2.18, and let I_1, I_2 be $\mathfrak{m}$-primary ideals. Then $DP(I_1) \leq DP(I_2)$ with equality if and only if $\overline{I_1} = \overline{I_2}$.*

Proof. This holds in much greater generality due to [4], Corollary 11. We need only that R is quasi-unmixed. □

2.3. Analytic Perspectives on the Log Canonical Threshold.

Definition 2.20. Let $0 \in \Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain (e.g. a ball). Let $\varphi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic (psh) function. The *log canonical threshold* of φ at 0 is given by

$$\operatorname{lct}(\varphi) = \sup\{s > 0 : e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } 0\}.$$

The fact that the invariants in Definitions 2.2 and 2.20 are both called the *log canonical threshold* is owed to the following proposition.

Proposition 2.21. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal primary to $(x_1, \ldots, x_n)$. Let $\underline{f} = f_1, \ldots, f_r$ be a generating set for $\mathfrak{a}$ and set $\varphi_{\underline{f}} = \log(|f_1|^2 + \cdots + |f_r|^2)$. Then $\varphi_{\underline{f}}$ is psh and $\operatorname{lct}(\mathfrak{a}) = \operatorname{lct}(\varphi_{\underline{f}})$.*

Proof. Follows from [20, Theorem 1.2]. □

Definition 2.22 ([7])**.** Let $0 \in \Omega \subseteq \mathbb{C}^n$ and let φ be a psh function on Ω which is locally bounded outside isolated singularities (or slightly more generally, see [op. cit.]). For $0 \leq j \leq n$, we define the *intersection numbers*

$$e_j(\varphi) = \int_{\{0\}} \left(\frac{i}{\pi}\partial\overline{\partial}\varphi\right)^j \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-j}.$$

The setup of Definition 2.22 is the original setting in which Demailly and Pham worked.

Theorem 2.23. *Let Ω, φ be as in Definition 2.22. Then $\operatorname{lct}(\varphi) = \infty$ if and only if $e_1(\varphi) = 0$ and, otherwise,*

$$c(\varphi) \geq \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)}.$$

Remark 2.24. When $\mathfrak{a}, \underline{f}, \varphi$ are as in Proposition 2.21, the mixed multiplicity $e_j(\mathfrak{a})$ coincides with the intersection number $e_j(\varphi_{\underline{f}})$. Applying the above result fo $\varphi = \varphi_{\underline{f}}$, one obtains the result $DP(\mathfrak{a}) \leq \operatorname{lct}(\mathfrak{a})$ of Theorem 1.1.

In [4], Bivià-Ausina considers the lct in the context of the ring of germs at 0 of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$.

Definition 2.25. Let $(\mathcal{O}_n, \mathfrak{m})$ denote the ring of germs at 0 of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$. If $\underline{f} = f_1, \ldots, f_r$ and $J = (\underline{f}) \subseteq \mathcal{O}_n$, then the lct of J is defined as

$$\operatorname{lct}(J) = \operatorname{lct}(\varphi_{\underline{f}}).$$

The numbers $e_j(J)$ can equivalently be defined as $e_j(\varphi_{\underline{f}})$ or as the mixed multiplicity $e(J^{\langle j\rangle}, \mathfrak{m}^{\langle n-j\rangle})$.

Biviá-Ausina's main result is the following.

Theorem 2.26 ([4], Theorem 13)**.** *Let $R = (\mathcal{O}_n, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. Fix coordinates $z_1, \ldots, z_n$ for $\mathcal{O}_n$. In the coordinates $z_1, \ldots, z_n$, let I^0 denote the smallest integrally-closed monomial ideal containing I. Then the following are equivalent:*

- *(i) There exist integers $d_1, \ldots, d_n$ such that $\overline{I} = \overline{(z_1^{d_1}, \ldots, z_n^{d_n})}$;*
- *(ii) $\operatorname{lct}(I^0) = DP(I)$;*
- *(iii) $\operatorname{lct}(I) = DP(I)$ and $\operatorname{lct}(I) = \operatorname{lct}(I^0)$.*

Remark 2.27. There is a bijection between $\mathfrak{m}$-primary ideals of $\mathcal{O}_n$ and $(x_1, \ldots, x_n)$-primary ideals of $\mathbb{C}[x_1, \ldots, x_n]$ which preserves both DP and lct, so the settings $\mathcal{O}_n, \mathbb{C}[x_1, \ldots, x_n]$ are equivalent in our context.

2.4. **Newton Polytopes of Monomial Ideals.** When working with monomial ideals, one often identifies a monomial $x_0^{a_0} \cdots x_n^{a_n}$ with the point $(a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. For future reference, it will help to give a name to this identification.

Definition 2.28. Let k be a field. We define the map

$$\log : \{\text{monomials in } k[x_0, \ldots, x_n]\} \to \mathbb{Z}_{\geq 0}^{n+1}, \qquad \log(x_0^{a_0} \cdots x_n^{a_n}) = (a_0, \ldots, a_n).$$

Definition 2.29. Let $\mathfrak{a} \subseteq k[x_0, \ldots, x_n]$ be a monomial ideal. Then the *Newton Polytope* of I, denoted $\Gamma(\mathfrak{a})$, is the convex hull in $\mathbb{R}^{n+1}$ of $\log(\mathfrak{a})$. Later on, we will let $\operatorname{conv}(-)$ denote the convex hull of a set.

Remark 2.30. We record several properties of $\Gamma(\mathfrak{a})$.

(i) $\Gamma(\mathfrak{a})$ is a closed, convex, unbounded subset of the first orthant of $\mathbb{R}^n$.
(ii) When $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the complement of $\Gamma(\mathfrak{a})$ inside the first orthant is an open, bounded polyhedron.
(iii) For two ideals $\mathfrak{a}, \mathfrak{b}$, the Minkowski sum of $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{b})$ is equal to $\Gamma(\mathfrak{a}\mathfrak{b})$. In particular, $\Gamma(\mathfrak{a}^n) = n\Gamma(\mathfrak{a})$.

Definition 2.31. Let $I \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal and $t \in \mathbb{Z}^+$. We let $[I]_t$ denote the vector space of t-forms in I.

The following proposition shows that Newton polytope of a monomial ideal determines the F-pure threshold.

Proposition 2.32. *Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$c(\mathfrak{a}) = \frac{1}{\mu}, \text{ where } \mu = \inf\{t : t\vec{1} \in \Gamma(\mathfrak{a})\}.$$

Proof. See [16], Example 5 for characteristic zero and [14], Proposition 36 for prime characteristic. □

Following the proof of [11], Theorem 1.4 and the terminology of [18], we also define the *limiting polytope* of a graded system of monomial ideals.

Definition 2.33. Let $\mathfrak{a}_\bullet$ be a graded system of monomial ideals. That is, suppose $\mathfrak{a}_r\mathfrak{a}_s \subseteq \mathfrak{a}_{r+s}$ for all $r, s \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}_\bullet)$ as the closure in $\mathbb{R}^{n+1}$ of the ascending union $\{\frac{1}{2^m}\Gamma(\mathfrak{a}_{2^m})\}_{m>0}$.

Definition 2.34. Let $>$ be a monomial order on R. We set $\Gamma_>(I) = \Gamma(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_n = \operatorname{in}_>(I^n)$.

2.5. **Integral Closure of Ideals.**

Definition 2.35. Let I be an ideal in a ring R. An element $r \in R$ is integral over I if there exists an integer n and elements $a_1, \ldots, a_n, a_i \in I^i$ such that

$$r^n + a_1 r^{n-1} + \cdots + a_n.$$

We then define the integral closure $\overline{I}$ of I as the set of elements $r \in R$ which are integral over I.

Those hoping for an exhaustive discussion of the integral closure of ideals should consult [17]. For now, we will list some basic properties of $\overline{I}$.

Proposition 2.36 (Properties of the Integral Closure, [17] Chapter 1)**.** *Let R be a ring and $I \subseteq R$ an ideal. Let $\varphi : R \to S$. Then we have*

(i): $\overline{I}$ *is an ideal.*
(ii): $\overline{(\overline{I})} = \overline{I}$.
(iii): $\overline{I}S \subseteq \overline{IS}$.
(iv): If $J \subseteq S$ is an ideal, then $\varphi^{-1}(\overline{J}) = \overline{\varphi^{-1}(J)}$.

(v): *For any multiplicatively-closed subset* $W \subseteq R$*, we have* $W^{-1}\overline{I} = \overline{W^{-1}I}$*.*

(vi): *The integral closure of a monomial ideal* $\mathfrak{a}$ *in a polynomial ring* $k[x_0, \ldots, x_n]$ *is generated by the set* $x^\alpha : \alpha \in \Gamma(\mathfrak{a})$*.*

(vii): *If* φ *is faithfully flat or an integral extension, then* $\overline{I}S \cap R = \overline{I}$*.*

Integral closure is an operation which respects many numerical invariants we are interested in this paper.

Theorem 2.37 ([17], Proposition 11.2.1, Theorem 11.3.1)**.** *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring and* $I \subseteq J$ *two* $\mathfrak{m}$*-primary ideals. Then* $e(I) = I(J)$ *if and only if* $\overline{I} = \overline{J}$*.*

The same result, of course, holds in the case that $(R, \mathfrak{m})$ is instead standard-graded.

Proposition 2.38. *Let* $I \subseteq k[x_1, \ldots, x_n]$ *be an ideal. Then* $c(I) = c(\overline{I})$*.*

Proof. For characteristic zero, see [20], Property 1.15. For positive characteristic, see [25], Proposition 2.2 (6). □

2.6. **Essential Dimension.**

Definition 2.39 (Essential Dimension)**.** Let $J \subseteq R = k[x_1, \ldots, x_d]$ be a homogeneous ideal. The essential dimension $\text{ess}(J)$ is equal to the minimal r for which there exist linear forms $\ell_1, \ldots, \ell_r$ such that J is extended from $I \subseteq k[\ell_1, \ldots, \ell_r]$.

We have the following result.

Proposition 2.40 ([1], Proposition 3.3)**.** *Let* k *be an algebraically-closed field,* $R = k[x_0, \ldots, x_n]$*, and* $J \subseteq R$ *a homogeneous ideal. Set* $r = \text{codim}(J)$*. Let* $L = (\ell_{r+1}, \ldots, \ell_n)$*, where the* ℓ_i *are chosen generally. For* $r \leq t \leq n$*, set* $L_t = (\ell_{t+1}, \ldots, \ell_n)$ *and* $J_t = \frac{J+L_t}{L_t}$*. Then for all* $r \leq t \leq n$*, we have* $ess(J_t) = \max(t + 1, ess(J))$*.*

3. The Limiting Polytope

3.1. **Complete Intersections in Positive Characteristic.** In this subsection, we prove [18, Theorem 1.1] over a field of characteristic $p > 0$. While the main argument is nearly identical, some intermediate lemmas must be weakened. In particular, [18, Lemma 3.6] is false in positive characteristic, which is evident by considering the generic initial ideal of $(x^p, y^p) \subseteq k[x, y]$ for an infinite field k of characteristic $p > 0$. For the sake of self-containedness, we will sketch the entire adapted argument here.

Lemma 3.1. *Let* k *be a field,* $R = k[x_1, \ldots, x_n]$*, and* $J \subseteq R$ *a homogeneous ideal. Let* $1 \leq j \leq n$ *and define* $\pi_j : R \to R/(x_{j+1}, \ldots, x_n) \cong k[x_1, \ldots, x_j]$*. If* $>$ *denotes the reverse lexicographic order, then*

$$\text{in}_> \pi_j(J) = \pi_j(\text{in}_>(J)).$$

Proof. Let $f \in J$ be a homogeneous element. Write $f = g + h$, where $h \in (x_{j+1}, \ldots, x_n)$ and $g \in k[x_1, \ldots, x_j]$. If $g = 0$, then $\pi_j(f) = 0$. If $g \neq 0$, then $\text{in}_>(f) = \text{in}_>(g)$. In both cases, we have $\pi_j(\text{in}_>(f)) = \text{in}_>(\pi_j(f))$. □

Definition 3.2. Let k be an infinite field. Let $R = k[x_1, \ldots, x_n]$ and let $>$ denote the reverse lexicographic order. Let $I = (f_1, \ldots, f_n)$ be a complete intersection ideal, where f_i is homogeneous of degree d_i and $d_1 \leq \cdots \leq d_n$. For $1 \leq j \leq n$, let $I_j := (f_1, \ldots, f_j)$. For $1 \leq j \leq n$, let $\pi_j : R \to R/(x_{j+1}, \ldots, x_n) \cong k[x_1, \ldots, x_j]$ denote the projection map. Let $\mathfrak{g}_m \in \text{GL}_n(k)$ be a general linear transformation such that $(\mathfrak{g}_m^{-1})(x_{j+1}, \ldots, x_n)$ is regular on R/I_j^m and $\text{in}_>(\mathfrak{g}_m I_j^m) = \text{gin}_>(I_j^m)$ for all $1 \leq j \leq n$.

Lemma 3.3. *Assume the setting of Definition 3.2. For all* $1 \leq j \leq n, m > 0$*, we have* $\text{gin}_>(I^m) = (\text{in}_>(\pi_j(\mathfrak{g}_m I_j^m)))R$*.*

Proof. Since I_j is a complete intersection, I_j^m is Cohen-Macaulay for all $m > 0$, hence $\operatorname{codim}(I_j^m) = \operatorname{depth}(I_j^m) = j$. Consequently, by [15, Lemma 3.1], the generators of $\operatorname{gin}_>(I_j^m)$ are contained in $k[x_1, \ldots, x_j]$, so $\pi_j(\operatorname{gin}_>(I_j^m))R = \operatorname{gin}_>(I_j^m)$. By Lemma 3.1, we have

$$\operatorname{gin}_>(I_j^m) = \pi_j(\operatorname{gin}_>(I_j^m))R = \pi_j(\operatorname{in}_>(\mathfrak{g}_m I_j^m))R = (\operatorname{in}_>(\pi_j(\mathfrak{g}_m I_j^m)))R.$$

□

The following is a general lemma which will be used repeatedly throughout the rest of this article.

Lemma 3.4. *Let L be a field, $S = L[x_1, \ldots, x_n]$, and $J \subseteq S$ an $\mathfrak{m}$-primary homogeneous ideal generated by forms of degree $\leq d$. Then $\mathfrak{m}^d \subseteq \overline{J}$.*

Proof. We first prove the result in the case that L is infinite. First, choose forms $f_1, \ldots, f_n$ from among the generators of J such that $(f_1, \ldots, f_n)$ is $\mathfrak{m}$-primary. If $h_1, \ldots, h_n$ are general linear forms, then

$$J' := (h_1^{d-\deg(f_1)} f_1, \ldots, h_n^{d-\deg(f_n)} f_n)$$

is an $\mathfrak{m}$-primary $(d, \ldots, d)$-complete intersection contained in J. As $J' \subseteq \mathfrak{m}^d$ and $e(J)' = d^n = e(\mathfrak{m}^d)$, we have $\mathfrak{m}^d = \overline{\mathfrak{m}^d} \subseteq \overline{J'} \subseteq \overline{J}$ by Theorem 2.37.

Now, let L be an arbitrary field, and set $S' = \overline{L}[x_1, \ldots, x_n]$. By Proposition 2.36 (vii) and the infinite field case, we have $\overline{J} = \overline{JS'} \cap S \supseteq \mathfrak{m}^d S' \cap S = \mathfrak{m}^d$. □

Lemma 3.5. *Assume the setting of Definition 3.2. Then for all $1 \leq j \leq n, m > 0$ we have $x_j^{(m+j-1)d_j} \in \operatorname{gin}_>(I_j^m)$.*

Proof. By Lemma 3.3, it suffices to prove the result when $j = n$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. By Lemma 3.4, we have $\mathfrak{m}^{d_n} \subseteq \overline{I}$. By the Briançon-Skoda theorem, we have $\mathfrak{m}^{(m+n-1)d_n} \subseteq \overline{I}^{m+n-1} \subseteq I^m$. It follows that

$$x_n^{(m+n-1)d_n} \in \mathfrak{m}^{(m+n-1)d_n} = \mathfrak{g}_m \mathfrak{m}^{(m+n-1)d_n} \subseteq \operatorname{in}_>(\mathfrak{g}_m I^m) = \operatorname{gin}_>(I^m).$$

□

Proposition 3.6. *Assume the setting of Definition 3.2. Let $\mathfrak{a}_\bullet$ be the graded system of ideals given by $\mathfrak{a}_m = \operatorname{gin}(I^m)$. Then*

$$\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, (d_1, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_n)\right). \tag{3}$$

Proof. By Lemma 3.5, we have

$$\operatorname{conv}\left(\vec{0}, (d_1, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_n)\right) \subseteq \overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}. \tag{4}$$

As $e(I) = d_1 \ldots d_n$, we also have $\operatorname{vol}(\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}) = (d_1 \ldots d_n)/n!$ by [21, Theorem 1.7]. It follows that the containment in Equation (4) is in fact equality. □

Corollary 3.7. *Assume the setup of Definition 3.2 and let $r < n$. Let $J = (f_1, \ldots, f_r)$ and for $m > 0$ set $\mathfrak{a}_m := \operatorname{gin}_>(J^m)$. Then we have*

$$\operatorname{conv}\left(\vec{0}, (d_1, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_r, 0, \ldots, 0)\right) = \overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}. \tag{5}$$

Proof. Follows from and Lemma 3.3 and Proposition 3.6. □

3.2. **F-Pure Thresholds and the Demailly-Pham Invariant.** In this subsection, we require an asymptotic version of Theorem 2.26 in arbitrary characteristic. Only minor refinements of Bivià-Ausina's arguments are needed.

Lemma 3.8. *Let k be an infinite field, $R = k[x_1, \ldots, x_n]$, and I an $\mathfrak{m}$-primary homogeneous ideal. If $>$ denotes the reverse lexicographic order, then for all $1 \leq j \leq n$ we have*

$$\lim_{t\to\infty} \frac{e_j(\mathrm{gin}_>(I^t))}{t^j} = e_j(I).$$

This result was shown by Bivià-Ausina [4, Theorem 4] in the related setting where $R = \mathcal{O}_n$ and where $>$ denotes the negative lexicographic order.

Proof. Without loss of generality, we first extend k to an uncountably infinite field; this changes neither the hypothesis nor the conclusion.

If $J \subseteq R$ is an $\mathfrak{m}$-primary homogeneous ideal and $h_1, \ldots, h_n$ is a sequence of linear forms, we say that a $h_1, \ldots, h_n$ *computes* $e_\bullet(J)$ if for $1 \leq j \leq n$, we have

$$e_j(J) = e\left(\frac{J + (h_1, \ldots, h_{n-j})}{(h_1, \ldots, h_{n-j})}\right).$$

For $m > 0$, we define:

- U_m is the open subset of $\mathrm{GL}_n(k)$ such that $\mathrm{in}_>(\mathfrak{g}I^m)$ is constant for all $\mathfrak{g} \in U_m$, such that U_m meets nontrivially the unipotent subgroup of upper triangular matrices with ones along the diagonal, and such that U is fixed by the Borel subgroup of upper-triangular matrices.
- V_m is the open subset of $\mathrm{GL}_n(k)$ for which $\mathfrak{g}x_n, \ldots, \mathfrak{g}x_1$ computes $e_\bullet(I^m)$.
- W_m is the open subset of $\mathrm{GL}_n(k)$ for which $\mathfrak{g}x_n, \ldots, \mathfrak{g}x_1$ computes $e_\bullet(\mathfrak{g}'(I^m))$, where $\mathfrak{g}' \in U_m$ is arbitrary.

Nonemptiness of U_m is [8, Theorem 15.18]. Nonemptiness of V_m, W_m follows from Proposition 2.17. Since k is uncountable, we may choose $\mathfrak{g} \in \bigcap_{m>0}(U_m \cap V_m \cap W_m)$. Set $J = (\mathfrak{g}^{-1})^* I$, and for $1 \leq j \leq n$, let $\pi_j : R \to R/(x_{j+1}, \ldots, x_n)$. We then have

$$\begin{aligned} e_j(I) = e(\pi_j(J)) &= \lim_{m\to\infty} \frac{1}{m^j} e(\pi_j(J^m)) \\ &= \lim_{m\to\infty} \frac{1}{m^j} e(\mathrm{in}_>(\pi_j(I^m))) = \lim_{m\to\infty} \frac{1}{m^j} e(\pi_j(\mathrm{in}_>(I^m))) \\ &= \lim_{m\to\infty} \frac{1}{m^j} e_j(\mathrm{in}_>(J^m)). \end{aligned}$$

The first and fifth equalities follow the fact that $x_1, \ldots, x_n$ computes $e_\bullet(J^m)$ and $e_\bullet(\mathrm{in}_>(J^m))$ for all $m \geq 1$. The second follows from the equality $e(\pi_j(J^m) = e(\pi_j(J)^m) = m^j e(\pi_j(J))$. The third is from [21, Corollary 1.13], and the fourth is from Lemma 3.1. ☐

Definition 3.9. Let k be an infinite field, $R = k[x_1, \ldots, x_n], \mathfrak{m} = (x_1, \ldots, x_n)$, and let $\mathfrak{a}_\bullet$ be a graded system of $\mathfrak{m}$-primary ideals. We define:

- The asymptotic mixed multiplicities: $e_j(\mathfrak{a}_\bullet) = \liminf_m \frac{e_j(\mathfrak{a}_m)}{m^j}$.
- The asymptotic Demailly-Pham invariant: $DP(\mathfrak{a}_\bullet) = \frac{1}{e_1(\mathfrak{a}_\bullet)} + \cdots + \frac{e_{n-1}(\mathfrak{a}_\bullet)}{e_n(\mathfrak{a}_\bullet)}$
- The asymptotic singularity threshold: $c(\mathfrak{a}_\bullet) = \liminf_m mc(\mathfrak{a}_m)$.

Before we prove our asymptotic version of Theorem 2.26, we require the following standard facts.

Lemma 3.10. *Let L be a field and $S = L[x_1, \ldots, x_n]$. Let I be a homogeneous ideal of S and $J \subseteq I$ denote the ideal of S generated by the homogeneous forms in I of degree $\leq d$. If $\mathrm{codim}(J) = n$, then $\overline{J} = \overline{I}$.*

Proof. It is clear that $\overline{J} \subseteq \overline{I}$. Let $\mathfrak{m} := (x_1, \ldots, x_n)$. For the reverse containment, note that $I \subseteq J + \mathfrak{m}^{d+1}$. By Lemma 3.4 we have

$$\overline{I} \subseteq \overline{J + \mathfrak{m}^{d+1}} \subseteq \overline{J + \mathfrak{m}^{d}} \subseteq \overline{J}.$$

□

Lemma 3.11. *Let L be a field and $S = L[x_1, \ldots, x_n]$. Let $J = (f_1, \ldots, f_n)$ be a complete intersection where $\deg f_i = d_i$ and $d_1 \leq \cdots \leq d_n$. Then we have the following:*

(i) If L is infinite, then for a general hyperplane section $H \subseteq \operatorname{Spec} R$, we have $e(I|_H) = d_1 \cdots d_{n-1}$.
(ii) With no assumption on $|L|$, we have $DP(I) = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$.

Proof. For (i), we note that for a general hyperplane section H, we have that $(f_1, \ldots, f_{n-1})|_H$ is $\mathfrak{m}$-primary. By Lemma 3.4, we have $(\mathfrak{m}|_H)^{d_{n-1}} \subseteq \overline{(f_1, \ldots, f_{n-1})|_H}$. As $f_n \in (\mathfrak{m}|_H)^{d_{n-1}}$, we have $\overline{(f_1, \ldots, f_{n-1})|_H} = \overline{J|_H}$. Consequently, $e(J|_H) = e(\overline{J|_H}) = d_1 \cdots d_{n-1}$.

For (ii), we note that $DP(J)$ is invariant under extension of the base field, so it suffices to consider the case of an infinite field. But then the result follows from (i) and Proposition 2.17 (iii). □

Corollary 3.12. *Let k be an infinite field, $R = k[x_1, \ldots, x_n]$, and I an $\mathfrak{m}$-primary homogeneous ideal. Then $DP(I) \leq c(I)$. Moreover, let $>$ denote the reverse lexicographic order. Suppose $DP(I) = c(I)$. Letting $\mathfrak{a}_m := \operatorname{gin}_>(I^m)$, we have*

$$\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, (e_1(I), 0, \ldots, 0), \left(0, \frac{e_2(I)}{e_1(I)}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \frac{e_n(I)}{e_{n-1(I)}}\right)\right). \tag{6}$$

Proof. By Lemma 3.8 we have $DP(\mathfrak{a}_\bullet) = DP(I)$ and by Proposition 2.10 we have $c(\mathfrak{a}_\bullet) \leq c(I)$. Let $\mu = \inf_t : (t, \ldots, t) \in \Gamma$ and set $\vec{\mu} = (\mu, \ldots, \mu)$. Since Γ is convex and $\mu \in \partial\Gamma$, by [23, Corollary 11.6.1] there exists a half-space $H^+ \subseteq \mathbb{R}^n$ such that $\Gamma \subseteq H^+$ and that $\mu \in \partial H^+$. Since Γ is closed under translation by elements of $R^n_{\geq 0}$ and the complement of Γ in $\mathbb{R}^n_{\geq 0}$ is bounded, the same is true for H^-. Consequently, the complement of H^- in $\mathbb{R}^n_{\geq 0}$ is a simplex which we denote by $\operatorname{conv}(0, (b_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, b_n))$.

Define a graded system of monomial ideals $\mathfrak{b}_\bullet$ by $\mathfrak{b}_m = \{x^\alpha : \alpha \in mH^+\}$. By assumption that $\Gamma \subseteq H^+$, we have $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all m. Consequently, we have $DP(\mathfrak{a}_\bullet) \leq DP(\mathfrak{b}_\bullet)$ by Proposition 2.19. By Proposition 2.32, we also have $c(\mathfrak{b}_\bullet) = c(\mathfrak{a}_\bullet)$. Altogether, we have

$$DP(I) = DP(\mathfrak{a}_\bullet) \leq DP(\mathfrak{b}_\bullet) = \frac{1}{b_1} + \cdots + \frac{1}{b_r} = c(\mathfrak{b}_\bullet) = c(\mathfrak{a}_\bullet) \leq c(I).$$

Now suppose $DP(I) = c(I)$. Then we also have $DP(\mathfrak{a}_\bullet) = DP(\mathfrak{b}_\bullet)$. By [4, Proposition 10], we further have that $e_j(I) = e_j(\mathfrak{a}_\bullet) = e_j(\mathfrak{b}_\bullet)$ for all $1 \leq j \leq n$. In particular, $e_n(\mathfrak{a}_\bullet) = e_n(\mathfrak{b}_\bullet)$, so by [21, Theorem 2.12 and Lemma 2.13], we have $\operatorname{vol}(\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)) = (\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{b}_\bullet))$. Since $\Gamma(\mathfrak{a}_\bullet), \Gamma(\mathfrak{b}_\bullet)$ are closed and convex with positive volume, it follows that $\Gamma(\mathfrak{a}_\bullet) = \Gamma(\mathfrak{b}_\bullet)$.

Since the generic initial ideal is Borel-fixed, we have $b_1 \leq \cdots \leq b_n$. Consequently, we can compute $e_j(\mathfrak{b}_\bullet)$ in terms of the numbers b_j: we have

$$\overline{\left(x_1^{\lfloor mb_1 \rfloor}, \ldots, x_n^{\lfloor mb_n \rfloor}\right)} \subseteq \mathfrak{b}_m \subseteq \overline{\left(x_1^{\lceil mb_1 \rceil}, \ldots, x_n^{\lceil mb_n \rceil}\right)}.$$

It follows that $e_j(\mathfrak{b}_\bullet) = b_1 \cdots b_j$. As $e_j(I) = e_j(\mathfrak{b}_\bullet)$, the result follows. □

Remark 3.13. The condition Equation (6) is necessary to have $c(I) = DP(I)$, but not sufficient. By [18] in characteristic zero or Proposition 3.6 in positive characteristic, Equation (6) holds for any homogeneous complete intersection $J = (f_1, \ldots, f_n)$.

3.3. **Behavior of the Threshold Under Modifications.** In this section, fix the following notation.

Definition 3.14. Let k be a characteristic zero field, $R = k[x_1, \dots, x_n]$, and let $\mathfrak{m}$ denote the homogeneous maximal ideal. Let $I \subseteq R$ be an $\mathfrak{m}$-primary homogeneous ideal. Write $I = I_1 + \cdots + I_r$, where I_j is generated by forms of degree d_j and $d_1 < \cdots < d_j$.

Let $A \subseteq k$ be a finitely-generated $\mathbb{Z}$-algebra and $J \subseteq A[x_1, \dots, x_n]$ an ideal such that $JR = I$. Such a subring A can always be constructed by adjoining to $\mathbb{Z}$ the field coefficients appearing in a generating set for I. If μ is a maximal ideal of A, we let I_μ denote the image of J in $(A/\mu)[x_1, \dots, x_n]$, and we write $I_\mu = I_{1,\mu} + \cdots + I_{r,\mu}$.

Lemma 3.15 ([2], Lemma 3.2)**.** *Let $R = k[x_1, \dots, x_n]$ and let $\mathfrak{m}$ denote the homogeneous maximal ideal. For any $e, t \in \mathbb{Z}^+$, we have*

$$(\mathfrak{m}^{[p^e]} : \mathfrak{m}^t) = \begin{cases} R & t \geq np^e - n + 1 \\ \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e - n + 1 - t} & t < np^e - n + 1 \end{cases}$$

More generally, we have the following.

Lemma 3.16. *Let $R = k[x_1, \dots, x_n]$. Let v be a monomial valuation on R with $v(x_i) \geq 0$ for all $1 \leq i \leq n$. For $\lambda \in \mathbb{R}^+$, let $\mathfrak{a}_\lambda$ denote the ideal $\{f \in R : v(f) \geq \lambda\}$ and $\mathfrak{a}_\lambda^+ = \{f \in R : v(f) > \lambda\}$. Let $q \in \mathbb{Z}^+, \lambda \in \mathbb{R}^+$. Then we have*

$$((x_1^q, \dots, x_n^q) : \mathfrak{a}_\lambda) = (x_1^q, \dots, x_n^q) + \mathfrak{a}^+_{(q-1)v(x_1 \dots x_n) - \lambda}. \tag{7}$$

Proof. The argument is the same as Lemma 3.15. Let $m \notin (x_1^q, \dots, x_n^q)$ be a monomial. Then $m \mid (x_1 \cdots x_n)^{q-1}$, so

$$\mathfrak{a}_\lambda m \not\subseteq (x_1^q, \dots, x_n^q) \iff \frac{(x_1 \dots x_b)^{q-1}}{m} \in \mathfrak{a}_\lambda \iff v((x_1 \cdots x_n)^{q-1}) - v(m) \leq \lambda.$$

We've shown that the two sides of Equation (7) contain the same monomials; both sides are monomial ideals, so the result follows. □

Lemma 3.17 ([1], Lemma 4.2)**.** *Let k be a field of characteristic $p > 0$, let $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ a homogeneous ideal. For a hyperplane H cut out by a linear form ℓ, we let $I|_H$ denote the image of I in $R/\ell R$. In this case, we have*

$$\nu_{I|_H}(p^e) \leq \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n-1)(p^e-1)+1}\}, \tag{8}$$

Corollary 3.18. *Assume the setup of Definition 3.14 and let $H \subseteq \operatorname{Spec} R$ be a hyperplane. Then $c(I) - c(I|_H) \geq 1/d_r$.*

For all $\mu \in \operatorname{Spec} A$, we have $c(I_\mu) - c(I_\mu|_{H_\mu}) \geq \frac{1}{d_r}$, and consequently, $c(I) - c(I|_H) \geq 1/d_r$.

Proof. We first prove the claim in characteristic $p > 0$. Combining Lemma 3.15 and Lemma 3.17, we have

$$\nu_{I|_H}(p^e) \leq \max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\}.$$

By Lemma 3.4, we have $\mathfrak{m}^{d_r} \subseteq \overline{I}$, so $\max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\} \leq \nu_I(p^e) - \left\lfloor \frac{p^e}{d_r} \right\rfloor$, so we have $\nu_{I|_H}(p^e) \leq \nu_I(p^e) - \left\lfloor \frac{p^e}{d} \right\rfloor$. Dividing by p^e and taking the limit as $e \to \infty$ gives the result.

In characteristic 0, we have

$$c(I) - c(I|_H) = \lim_{\substack{\mu \in \operatorname{Spec} A \\ \operatorname{char} A/\mu \to \infty}} c(I|_\mu) - c(I_\mu|_{H_\mu}) \geq 1/d_r.$$

□

Lemma 3.19. *Assume the setup of Definition 3.14. Suppose $r = 2$. Then we have $c(I) = \frac{n}{d_2} + c(I_1)\frac{d_2-d_1}{d_2}$. In particular, for any $0 \le s \le n$, we have $c(I) = \frac{s}{d_1} + \frac{n-s}{d_2}$ if and only if $c(I_1) = \frac{s}{d_1}$.*

Proof. We prove the claim first in positive characteristic. By Lemma 3.4, we have $\mathfrak{m}^{d_2} \subseteq \overline{I}$, so $I \subseteq I_1 + \mathfrak{m}^{d_2} \subseteq \overline{I}$, so $c(I) = c(I_1 + \mathfrak{m}^{d_2})$. Consequently, we have

$$\nu_{\overline{I}}(p^e) = \max\left\{a + b : I_1^a \mathfrak{m}_\mu^{bd_n} \not\subseteq \mathfrak{m}^{[p^e]}\right\} = \max\{a + b : I_1^a \not\subseteq (\mathfrak{m}_p^{[p^e]} : \mathfrak{m}_p^{bd_n})\}.$$

By Lemma 3.15, this is equivalent to

$$\nu_{\overline{I}}(p^e) = \max\{a + b : I_1^a \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e-n+1-bd_2}\} = \max_{0 \le a \le \nu_{I_1}(p^e)} a + \frac{np^e - n + 1 - ad_1}{d_2}. \tag{9}$$

The quantity being maximized in Equation (9) is an increasing function of a, so the maximum occurs at $a = \nu_{I_1}(p^e)$ and

$$\nu_{\overline{I}}(p^e) = \frac{np^e - n + 1}{d_2} + \nu_{I_1}(p^e)\frac{d_2 - d_1}{d_2}.$$

Dividing by p^e and letting $e \to \infty$, we obtain

$$c(I) = \frac{n}{d_2} + c(I_1)\frac{d_2 - d_1}{d_2}.$$

For characteristic zero, we compute

$$c(I) = \sup_\mu c(I_\mu) = \sup_\mu \left(\frac{n}{d_2} + c(I_{1,\mu})\frac{d_2 - d_1}{d_2}\right) = \frac{n}{d_2} + c(I)\frac{d_2 - d_1}{d_2}.$$

□

The final lemma of this section is a trivial combination of facts from the preliminary section, but we will use it repeatedly and so we state it here.

Lemma 3.20. *Let $I \subseteq R$ be an ideal and $>$ a monomial partial order. If $J \subseteq \mathrm{in}_>(\overline{I})$, then $c(J) \le c(I)$.*

Proof. Follows from Propositions 2.9 and 2.10. □

4. Proof of Theorem 4.22 in the Complete Intersection Case

Assumption 4.1. *We assume a setup similar to Definition 3.14. Let k be an algebraically-closed field. Let $a_1, \dots, a_r, d_1, \dots, d_r \in \mathbb{Z}^+$. For $1 \le i \le r$, let $\mathbf{x}_i$ denote the tuple of variables $x_{i,1}, \dots, x_{i,a_i}$, and let $R = k[\mathbf{x}_1, \dots, \mathbf{x}_r]$. Let $I \subseteq R$ be a complete intersection of the form $(f_{1,1}, \dots, f_{r,a_r})$ such that $f_{i,j}$ is a d_i-form. For $1 \le j \le r$, write $I_j = (f_{j,1}, \dots, f_{j,a_j})$. Let v denote the monomial valuation with $v(x_{i,j}) = 1/d_i$. We let $\mathfrak{D}$ denote the ideal $\overline{(\mathbf{x}_1^{d_1}, \dots, \mathbf{x}_r^{d_r})}$, which coincides with the set of elements of valuation $v(-) \ge 1$.*

Assumption 4.2. *Assume the setup of Assumption 4.1. We define the following condition on the ideal I:*

$$I_1 \text{ is extended from } k[\mathbf{x}_1] \text{ and } I \subseteq \mathfrak{D} + (\mathbf{x}_1). \tag{10}$$

Definition 4.3. For $r \in \mathbb{Z}^+$, we define the statements A_r, B_r.

(A_r) For all I as in Assumption 4.1, if $DP(I) = c(I)$ then there exists $\mathfrak{g} \in \mathrm{GL}_n(k)$ such that $\mathfrak{g}I \subseteq \mathfrak{D}$.

(B_r) For all I as in Assumption 4.1, if $DP(I) = c(I)$ then there exists $\mathfrak{g} \in \mathrm{GL}_n(k)$ such that $\mathfrak{g}I$ satisfies Equation (10).

The goal of this section is to prove A_r for all r. We accomplish this via the following steps:

(1) A_1, A_2 hold
(2) For $r \geq 3$, $A_2, A_{r-1} \implies B_r$
(3) For $r \geq 3$, A_{r-1}, B_r together imply A_r.

4.1. **Step (1): A_1, A_2 hold.**

Proof of A_1. If I is an $\mathfrak{m}$-primary complete intersection generated by forms of degree d for some d, by Lemma 3.4 we have $\overline{I} = \mathfrak{m}^d$. □

When $r = 1$, the hypothesis $c(I) = DP(I)$ is satisfied for all choices of I, so we are not able to use A_1 as a useful base case.

Proof of A_2. By Lemma 3.19, if $c(I) = DP(I)$ then $c(I_1) = a_1/d_1$. By [10, Theorem 3.5] in characteristic zero and Theorem 1.2 in positive characteristic, it follows that $\operatorname{ess}(I_1) = a_1$. By [1, Lemma 3.18], there exists $\mathfrak{g} \in \mathrm{GL}_n(k)$ such that $\overline{\mathfrak{g}(I_1)} = (\mathbf{x}_1)^{d_1}$, hence $\mathfrak{g}I \subseteq (\mathbf{x}_1)^{d_1} + \mathfrak{m}^{d_2} \subseteq \mathfrak{D}$. □

4.2. **Step (2): For $r \geq 3$, $A_2, A_{r-1} \implies B_r$.**

Lemma 4.4. *Assume the setup of Assumption 4.1 and suppose $c(I) = DP(I)$. Let $\ell \in R$ be a general linear form and let H denote the zero locus of ℓ. Then $c(H, I|_H) = DP(I|_H)$.*

Proof. By Lemma 3.11, Corollary 3.12, and Corollary 3.18, we have

$$\frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r} = DP(I|_H) \leq c(I|_H) \leq \frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r}.$$

□

Lemma 4.5. *Assume the setup of Assumption 4.1 and suppose $r \geq 3, c(I) = DP(I)$. If A_{r-1} holds, there exists $\mathfrak{g} \in \mathrm{GL}_n(k)$ such that $\mathfrak{g}\overline{I_1} = (\mathbf{x}_1)^{d_1}$.*

Proof. Note that $DP(I) = a_1/d_1 + \cdots + a_r/d_r$. Let L be an ideal of R generated by $a_3 + \cdots + a_d$ general linear forms. Since I is a complete intersection, $I_1 + I_2 + L$ is $\frac{\mathfrak{m}}{L}$-primary, so by Lemma 3.10 we have $c(R/L, \frac{I+L}{L}) = c(R/L, \frac{I_1+I_2+L}{L})$. Consequently, by repeated application of Corollary 3.18, we have

$$c(I) \geq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) + \frac{a_3}{d_3} + \cdots + \frac{a_r}{d_r}. \tag{11}$$

Assuming $c(I) = DP(I)$, we have

$$\frac{a_1}{d_1} + \frac{a_2}{d_2} \leq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) \leq \frac{a_1}{d_1} + \frac{a_2}{d_2},$$

where the left-hand side is by Corollary 3.12 and the right-hand side is by Equation (11). Both inequalities are therefore equalities, so by the argument of A_2, we have $\operatorname{ess}\left(\frac{I_1+L}{L}\right) = a_1$. By Proposition 2.40, it follows that $\operatorname{ess}(I_1) = a_1$. The result then follows from [1, Lemma 3.18]. □

Lemma 4.6. *Assume the setup of Assumption 4.1. Suppose $c(I) = DP(I)$. Then there exists $\mathfrak{g} \in \mathrm{GL}_n(k)$ such that $\mathfrak{g}I$ satisfies Equation (10).*

Proof. By Lemma 4.5, we may assume I_1 is extended from $k[\mathbf{x}_1]$. Let $\succ$ denote the monomial partial order induced by the monomial valuation $w(x_{1,i}) = 0$ and $w(x_{i,j}) = 1$ for $i \geq 2$. For $2 \leq i \leq r, 1 \leq j \leq a_i$, let $g_{i,j} := \operatorname{in}_\succ(f_{i,j})$. Since I is a complete intersection, we have $f_{i,j} \notin \sqrt{I_1} = (\mathbf{x}_1)$, hence $g_{i,j} \notin (\mathbf{x}_1)$ and moreover $f_{i,j} - g_{i,j} \in (\mathbf{x}_1)$. Observe that

$$\operatorname{in}_\succ(I) \supseteq I_1 + \operatorname{in}_\succ(I_2 + \cdots + I_r) \supseteq I_1 + (g_{2,1}, \ldots, g_{r,a_r}). \tag{12}$$

Let I' denote the right-hand side of Equation (12). Because $g_{i,j}$ and $f_{i,j}$ have the same image modulo $(\mathbf{x}) = \sqrt{I_1}$, the ideal I' is a complete intersection. In particular, I' is a complete intersection of type $(\underbrace{d_1, \ldots, d_1}_{a_1}, \ldots, \underbrace{d_r, \ldots, d_r}_{a_r})$. By Lemma 3.11 and Proposition 2.10, we have

$$DP(I) = DP(I') \leq c(I') \leq c(\mathrm{in}_{\succ}(\varphi^* I)) \leq c(I) = DP(I). \tag{13}$$

As $\overline{I_1} = (\mathbf{x}_1)^{d_1}$, we have $c(I_1) = a_1/d_1$. Since I_1 and $(g_{2,1}, \ldots, g_{r,a_r})$ are defined in terms of disjoint sets of variables, we have by [26], Theorem 2.4 (1) that

$$c(R, I') = c(k[\mathbf{x}_1], I_1) + c(k[\mathbf{x}_2, \ldots, \mathbf{x}_r], (g_{2,1}, \ldots, g_{r,a_r})) = \frac{a_1}{d_1} + c(k[\mathbf{x}_2, \ldots, \mathbf{x}_r], (g_{2,1}, \ldots, g_{r,a_r})). \tag{14}$$

It follows from Equations (13) and (14) that $(g_{2,1}, \ldots, g_{r,a_r})$, which is a complete intersection in $k[\mathbf{x}_2, \ldots, \mathbf{x}_r]$, also has $DP = c$. By A_{r-1}, there exists $\mathfrak{g} \in \mathrm{GL}_{n-a_1}(k)$ such that $\mathfrak{g}(g_{2,1}, \ldots, g_{r-1,a_{r-1}}) \subseteq \overline{(\mathbf{x}_2^{d_2}, \ldots, \mathbf{x}_r^{d_r})}$.

We define $\mathfrak{g}' := \begin{bmatrix} \mathrm{id}_{a_1} & 0 \\ 0 & \mathfrak{g} \end{bmatrix}$ and we claim that $\mathfrak{g}' I$ satisfies Equation (10). By construction, $\mathfrak{g}' I'$ satisfies Equation (10). Since $g_{i,j} - f_{i,j} \in (\mathbf{x}_1)$ for all $2 \leq i \leq r, 1 \leq j \leq a_i$, we have

$$\mathfrak{g}'(I_2 + \cdots + I_r) \subseteq \mathfrak{g}'(g_{1,1}, \ldots, g_{r,a_r}) + (\mathbf{x}_1),$$

which proves that $\mathfrak{g}' I$ also satisfies Equation (10). □

4.3. **Step (3):** $A_{r-1}, B_r \implies A_r$.

Definition 4.7. If $f = \sum_{b \in \mathbb{Z}^n} \gamma_b x^b \in R$, we define $\mathrm{supp}(f) = \{x^b : \gamma_b \neq 0\}$.

Lemma 4.8. *Assume the setting of Assumption 4.1. Suppose I satisfies Equation* (10) *and $I \not\subseteq \mathfrak{D}$. Then there exists an ideal J and an integer $2 \leq m \leq r-1$ such that:*

- ***(C.i)*** *There exist homogeneous d_m-forms $h_{m,1}, \ldots, h_{m,a_m}$ such that*

$$\overline{J} = \overline{(\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (h_{m,1}, \ldots, h_{m,a_m}) + (\mathbf{x}_{m+1})^{d_{m+1}} + \cdots + (\mathbf{x}_r)^{d_r}}. \tag{15}$$

- ***(C.ii)*** *For all $1 \leq j \leq a_m$, we have $h_{m,j} = h'_{m,j} + h''_{m,j}$, where $h'_{m,j} \in k[\mathbf{x}_m], h''_{m,j} \in (\mathbf{x}_1)$, and $\mathrm{supp}(h''_{m,j}) \cap \mathfrak{D} = \varnothing$. Moreover, there exists some j such that $h_{m,j} - h'_{m,j} \neq 0$;*
- ***(C.iii)*** *$c(J) \leq c(I)$.*

In the process of constructing J, we will construct a sequence of ideals $J_0, \ldots, J_s$.

Notation 4.9. *Recall that $n = a_1 + \cdots + a_r$. For $1 \leq i \leq r$, we define the map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ by*

$$\pi_i((b_{1,1}, \ldots, b_{r,a_r})) = b_{i,1} + \cdots + b_{i,a_i}.$$

Moreover, we define the map $\pi : \mathbb{R}^n \to \mathbb{R}^r$ by $\pi(b) = (\pi_1(b), \ldots, \pi_r(b))$. If J_k is given by

$$J_k = (\mathbf{x}_1)^d + (g_{2,1,k}, \ldots, g_{r-1,a_{r-1},k}) + \mathfrak{m}^{d_r},$$

we define

$$T_k^{i,j} := \left\{ \pi(b) : x^b \in \mathrm{supp}(g_{i,j,k}) : \pi_1(b) > 0 \right\}, \quad T_k^i = \bigcup_{j=1}^{a_i} T_k^{i,j}, \quad T_k = \bigcup_{i=2}^{r-1} T_k^i. \tag{16}$$

Additionally, we define auxiliary conditions on each J_k. For $k = 0$, we define the conditions (D.i)-(D.iv).

- **(D.i)** For all $2 \leq i \leq r-1, 1 \leq j \leq a_i$, we have $\mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i] \subseteq \mathrm{supp}(g_{i,j,0}) \subseteq \mathrm{supp}(f_{i,j})$;
- **(D.ii)** We have $T_0 \neq \varnothing$;
- **(D.iii)** For all x^b such that $(u_1, \ldots, u_r) \in T_0$, we have $\sum_{i=1}^r \frac{u_i}{d_i} < 1$;
- **(D.iv)** $c(J_0) \leq c(I)$.

For $k \geq 1$, we define the conditions (E.i)-(E.iv).

(E.i) For all $2 \leq i \leq r-1, 1 \leq j \leq a_i$, we have $\operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i] = \operatorname{supp}(g_{i,j,k}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]$ and $\operatorname{supp}(g_{i,j,k}) \subseteq \operatorname{supp}(g_{i,j,k-1})$;

(E.ii) We have $T_k \neq \varnothing$;

(E.iii) For all x^b such that $\pi(b) \in T_k$, we have $\sum_{i=1}^r \frac{\pi_i(b)}{d_i} < 1$;

(E.iv) $c(J_k) \leq c(J_{k-1})$.

Before we begin the proof of Lemma 4.8, we state two lemmas.

Lemma 4.10. *Assume the setting of Assumption 4.1. Suppose I satisfies Equation* (10)*. Then for all $2 \leq i \leq r-1, 1 \leq j \leq a_i$, we have* $\operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i] \neq \varnothing$.

Proof. Since I satisfies Equation (10), by Proposition 2.19 we have $\overline{\frac{I_2+\cdots+I_r+(\mathbf{x}_1)}{(\mathbf{x}_1)}} = \frac{\mathfrak{D}+(\mathbf{x}_1)}{(\mathbf{x}_1)}$. It follows that $\overline{\frac{I_2+(\mathbf{x}_1)}{(\mathbf{x}_1)}} = (\mathbf{x}_2)^{d_2}$. In particular, we have that $f_{2,j} \in k[\mathbf{x}_2] + (\mathbf{x}_1)$. As $f_{2,1}, \dots, f_{2,a_2}$ is a complete intersection mod $(\mathbf{x}_1)$, we conclude that $\operatorname{supp}(f_{2,j}) \cap k[\mathbf{x}_2] \neq 0$. For $3 \leq i \leq r-1$, we apply the same argument to the image of $f_{i,j}$ mod $(\mathbf{x}_1, \dots, \mathbf{x}_{i-1})$, proving the claim. ☐

Lemma 4.11. *Let R be as in Assumption 4.1. Suppose that $f_1, \dots, f_{a_i}$ are d_i-forms comprising a regular sequence in R, and suppose that $f_j \in k[\mathbf{x}_i]$ for all $1 \leq j \leq a_i$. Then the integral closure J of $(f_1, \dots, f_j)$ in R is equal to $(\mathbf{x}_i)^d$.*

Proof. Since $k[\mathbf{x}_i] \to R$ is faithfully flat, $f_1, \dots, f_{a_i}$ form a regular sequence in $k[\mathbf{x}_i]$. By Theorem 2.37, the integral closure of $(f_1, \dots, f_{a_i})$ in $k[\mathbf{x}_i]$ is $(\mathbf{x}_i)^d$. By [17, Proposition 1.6.2], we have $(\mathbf{x}_i)^{d_i} \subseteq J$. On the other hand, we have $(f_1, \dots, f_{a_i}) \subseteq (\mathbf{x}_i)^{d_i}$. By [17, Proposition 1.4.6], $(\mathbf{x}_i)^d$ is integrally closed in R, so $J = (\mathbf{x}_i)^{d_i}$. ☐

Lemma 4.12. *Assume the setting of Assumption 4.1. Suppose I satisfies Equation* (10) *and $I \not\subseteq \mathfrak{D}$. Then there exists an ideal J_0 satisfying conditions (D.i)-(D.iv) of Notation 4.9.*

Proof. We define a set T_I which measures the failure of I to be contained in $\mathfrak{D}$. We define the sets

$$S_I^{i,j} := \left\{\pi(b) : x^b \in \operatorname{supp}(f_{i,j}), \pi_1(b) \neq 0\right\}, \quad S_0^i = \bigcup_{j=1}^{a_i} S_I^{i,j}, \quad S_I = \bigcup_{i=2}^{r-1} S_I^i. \tag{17}$$

Note that the condition $x^b \notin \mathfrak{D}$ is equivalent to the condition $v(x^b) < 1$, so we define

$$T_I^{i,j} = \{(u_1, \dots, u_r) \in S_I^{i,j} : \sum_{i=1}^r \frac{u_i}{d_i} < 1\}, \quad T_I^i = \bigcup_{j=1}^{a_i} T_I^{i,j}, \quad T_I = \bigcup_{i=2}^{r-1} T_I^i. \tag{18}$$

Since I satisfies Equation (10), we also have $u_1 > 0$ for all $(u_1, \dots, u_r) \in S_I$. Let

$$t_0 := \max_{(u_1,\dots,u_r)\in S_I} \frac{1 - u_1/d_1 - \cdots - u_r/d_r}{u_1}.$$

Since $I \not\subseteq \mathfrak{D}$, we have $t_0 > 0$ and the elements of S_I achieving this maximum are all in T_I. Define $w_0 : \mathbb{Z}^r \to \mathbb{Q}$ by

$$w_0(u_1, \dots, u_r) = \left(-\frac{1}{d_1} - t_0\right) u_1 - \frac{u_2}{d_2} - \cdots - \frac{u_r}{d_r}.$$

For any $x^b \in \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]$, we have $w_0(\pi(b)) = -v(x^b)$. Since I satisfies Equation (10), for any $x^b \in \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]$ we have $w_0(\pi(b)) \leq -1$.

By Lemma 4.10, $\operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i]$ is nonempty for all $2 \leq i \leq r-1, 1 \leq j \leq a_i$. For any $x^b \in \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i]$, we have $w_0(\pi(b)) = -v(x^b) = -1$. Consequently, for all $2 \leq i \leq r-1, 1 \leq j \leq a_i$ we have

$$\max_{x^b \in \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_2,\dots,\mathbf{x}_r]} w_0(\pi(b)) = -1. \tag{19}$$

For all $2 \leq i \leq r, (u_1, \dots, u_r) \in S_I^i$ we have

$$-1 = \left(-\frac{1}{d_1} - \frac{1 - u_1/d_1 - \cdots - u_r/d_r}{u_1}\right) u_1 - \frac{u_2}{d_2} - \cdots - \frac{u_r}{d_r} \geq w_0(u_1, \dots, u_r). \tag{20}$$

We define a monomial partial order $>_0$ by

$$x^b >_0 x^{b'} \iff w_0(\pi(b)) > w_0(\pi(b')). \tag{21}$$

For $2 \leq i \leq r-1, 1 \leq j \leq a_i$, we set $g_{i,j,0} = \mathrm{in}_{>_0}(f_{i,j})$. We define

$$J_0 = (\mathbf{x}_1)^{d_1} + (g_{2,1,0}, \dots, g_{r-1,a_{r-1},0}) + \mathfrak{m}^{d_r}.$$

By Equations (19) and (20), for $2 \leq i \leq r-1$ we have

$$\mathrm{supp}(\mathrm{in}_{>_0}(f_{i,j})) = \{x^b \in \mathrm{supp}(f_{i,j}) : w_0(b) = -1\}. \tag{22}$$

We now verify that J_0 satisfies the conditions (D.i)-(D.iv).

(D.i) By Equation (22), we have $\mathrm{supp}(g_{i,j,0}) \subseteq \mathrm{supp}(f_{i,j})$. By Equations (19) and (20), we also have $\mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i] \subseteq \mathrm{supp}(g_{i,j,0})$.

(D.ii) We constructed w_0 so that Equation (20) is sharp for some $u \in T_I$.

(D.iii) For all $(u_1, \dots, u_r) \in S_I \setminus T_I$ we have

$$w_0(u_1, \dots, u_r) = -t_0 u_1 - \sum_{i=1}^{r} \frac{u_i}{d_i} \leq t_0 u_1 - 1 < -1.$$

(D.iv) Follows from Lemma 3.20.

□

Lemma 4.13. *Assume the setting of Assumption 4.1 and Lemma 4.12. Then there exists an ideal J_1 satisfying conditions (E.i)-(E.iv) of Notation 4.9.*

Proof. To begin, we set

$$t_1 := \max_{2 \leq i \leq r-1} \max_{(u_1,\dots,u_r) \in T_1^i} \frac{d_r^2 u_2 + \cdots + d_r^r u_r - d_i d_r^i}{u_1}, \quad w_1(u_1, \dots, u_r) := t_1 u_1 - d_r^2 u_2 - \cdots - d_r^r u_r. \tag{23}$$

Analogously to Equation (21), we use w_1 to define a monomial partial order $>_1$. Next, given $2 \leq i \leq r-1, 1 \leq j \leq a_i$, we set $g_{i,j,1} = \mathrm{in}_{>_1}(g_{i,j,0})$ and we define

$$J_1 = (\mathbf{x}_1)^{d_1} + (g_{2,1,1}, \dots, g_{r-1,a_{r-1},1}) + \mathfrak{m}^{d_r}.$$

We now verify conditions (E.i)-(E.iv).

(E.i) The containment $\mathrm{supp}(g_{i,j,1}) \subseteq \mathrm{supp}(g_{i,j,0})$ is immediate. For $2 \leq i \leq r-1, u \in T_1^i$, we have $w_1(u) \leq -d_i d_r^i$. Let $x^b \in \mathrm{supp}(g_{i,j,0}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]$. If $x^b \in k[\mathbf{x}_i]$, then $w_1(\pi(b)) = -d_i d_r^i$. If $x^b \notin k[\mathbf{x}_i]$, then since $v(x^b) = 1$, there exists some $i < l \leq r, 1 \leq j \leq a_l$ such that $x_{l,j} \mid x^b$, hence

$$w_1(x^b) \leq w(x_{l,j}) \leq -d_r^{i+1} < -d_i d_r^i. \tag{24}$$

It follows that

$$\mathrm{supp}(g_{i,j,1}) = \{x^b \in \mathrm{supp}(g_{i,j,0} : w_1(\pi(b)) = -d_i d_r^i)\}. \tag{25}$$

(E.ii) Let $2 \leq i \leq r-1, u \in T_1^{i,j}$ such that u realizes the maximum in Equation (23). Then $w_1(u) = -d_i d_r^i$, so the monomial summand $x^b \in \mathrm{supp}(g_{i,j,0})$ such that $\pi(b) = u$ ties for the leading term of $g_{i,j,0}$, hence $T_1^{i,j} \neq \varnothing$.

(E.iii) Follows from the fact that $\mathrm{supp}(g_{i,j,1}) \subseteq \mathrm{supp}(g_{i,j,0})$ and J_0 satisfies (iii).

(E.iv) Follows from Lemma 3.20.

□

We may now prove Lemma 4.8.

Proof. Let J_0, J_1 be as in Lemmas 4.12 and 4.13. Let $k \geq 1$ and assume that:

- We have constructed $J_0, \dots, J_k$;
- J_0 satisfies (D.i)-(D.iv);
- $J_1, \dots, J_k$ satisfy (E.i)-(E.iv).

Let $\Lambda_k = \{2 \leq i \leq r-1 : T_k^i \neq \varnothing\}$. By assumption, $|\Lambda_k| \neq \varnothing$. If $|\Lambda_k| = 1$, then we set $s = k$ so that J_k will be our final ideal in the sequence $J_0, \dots, J_s$. Otherwise, we will construct an ideal J_{k+1} such that $\varnothing \subsetneq \Lambda_{k+1} \subsetneq \Lambda_k$. Let $\lambda_k = \min \Lambda_k$. Let

$$t_{k+1} := \max_{i \in \Lambda_k \setminus \{\lambda_k\}, (u_1, \dots, u_r) \in T_k^i} \frac{u_{\lambda_k}}{u_1}, \quad w_{k+1}(u_1, \dots, u_r) := -t_{k+1} u_1 + u_{\lambda_k}. \tag{26}$$

Analogously to Equation (21), we use w_{k+1} to define a partial order $>_{k+1}$, and we set $g_{i,j,k+1} = \mathrm{in}_>(g_{i,j,k})$. We then define

$$J_{k+1} = (\mathbf{x}_1)^{d_1} + (g_{2,1,k+1}, \dots, g_{r-1,a_{r-1},k+1}) + \mathfrak{m}^{d_r}.$$

For all $1 \leq j \leq a_{\lambda_k}$ and all $x^b \in \mathrm{supp}(g_{\lambda_k,j,k})$, setting $u = \pi(b)$ we have

$$w_{k+1}(u) \leq u_{\lambda_k} \leq d_k. \tag{27}$$

By Lemma 4.10 and Equation (27), we conclude that

$$\mathrm{supp}(g_{\lambda_k,j,k+1}) = \{x^b \in \mathrm{supp}(g_{\lambda_k,j,k}) : w_k(\pi(b)) = d_k\} = \mathrm{supp}(g_{\lambda_k,j,k}) \cap k[\mathbf{x}_{\lambda_k}]. \tag{28}$$

Let $2 \leq i \leq r, i \neq \lambda_k$ and $1 \leq j \leq a_i$. If $x^b \in \mathrm{supp}(g_{i,j,k})$, setting $u = \pi(b)$ we have

$$w_{k+1}(u) \leq \left(-\frac{u_{\lambda_k}}{u_1}\right) u_1 + u_{\lambda_k} = 0.$$

By Lemma 4.10, there exists $x^b \in \mathrm{supp}(g_{i,j,k}) \cap k[\mathbf{x}_i]$, so we have

$$\mathrm{supp}(g_{i,j,k+1}) = \{x^b \in \mathrm{supp}(g_{i,j,k}) : w_{k+1}(\pi(b)) = 0\} \supseteq \mathrm{supp}(g_{i,j,k}) \cap k[\mathbf{x}_i]. \tag{29}$$

We show that J_{k+1} satisfies (E.i)-(E.iv).

(E.i) For all $2 \leq i \leq r-1$, the containment $\mathrm{supp}(g_{i,j,k+1}) \subseteq \mathrm{supp}(g_{i,j,k})$ follows from the construction of $g_{i,j,k+1}$ as an initial term of $g_{i,j,k}$. Since J_k satisfies (E.i), we note that

$$\mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i] = (\mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i]) \cap k[\mathbf{x}_i] = (\mathrm{supp}(g_{i,j,k}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]) \cap k[\mathbf{x}_i] = \mathrm{supp}(g_{i,j,k}) \cap k[\mathbf{x}_i]. \tag{30}$$

For $i \neq \lambda_k$, the containment $\mathrm{supp}(g_{i,j,k+1}) \subseteq \mathrm{supp}(f_{i,j})$ together with Equations (29) and (30) imply

$$\mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i] \supseteq \mathrm{supp}(g_{i,j,k+1} \cap k[\mathbf{x}_i]) \supseteq (g_{i,j,k}) \cap k[\mathbf{x}_i] = \mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_i].$$

For $i = \lambda_k$, by Equation (28) we have $\mathrm{supp}(g_{\lambda_k,j,k+1}) \cap k[\mathbf{x}_{\lambda_k}] = \mathrm{supp}(g_{i,j,k+1}) = \mathrm{supp}(f_{i,j}) \cap k[\mathbf{x}_{\lambda_k}]$.

(E.ii) Let $2 \leq i \leq r-1, i \neq \lambda_k, u \in T_k^{i,j}$ such that u realizes the maximum in Equation (26). Then $w_{k+1}(u) = 0$, so the monomial summand $x^b \in \mathrm{supp}(g_{i,j,k})$ such that $\pi(b) = u$ ties for the leading term of $g_{i,j,k}$, hence $T_{k+1}^{i,j} \neq \varnothing$.

(E.iii) Follows from (E.iii) for J_k and (E.i) for J_{k+1}.

(E.iv) Follows from Lemma 3.20.

Since $|\Lambda_k| \leq r-2$, we verify that the final ideal J_s in the sequence $J_0, \dots, J_s$ satisfies conditions (C.i)-(C.iii) where m is the unique element of Λ_s. For all $2 \leq i \leq r-1, 1 \leq j \leq a_i$, set $h_{i,j} := g_{i,j,s}$. Given $h_{i,j} = \sum_{x^b \in \mathrm{supp}(h_{i,j})} \gamma_b x^b$, we set

$$h'_{i,j} = \sum_{x^b \in \mathrm{supp}(h_{i,j}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r]} \gamma_b x^b.$$

By (E.i), we note that $f_{i,j} - h'_{i,j} \in (\mathbf{x}_1)$.

(C.i) By Lemma 4.11, we have $\overline{I_1} = (\mathbf{x}_1)^{d_1}$, hence $\sqrt{I_1} = (\mathbf{x}_1)$. As I is a complete intersection, it follows that for all $2 \leq i \leq r-1$ we have

$$a_1 + a_i = \operatorname{codim}(I_1 + (f_{i,1}, \dots, f_{i,a_i})) = \operatorname{codim}((\mathbf{x}_1) + (f_{i,1}, \dots, f_{i,a_i})) = \operatorname{codim}((\mathbf{x}_1) + (h'_{i,1}, \dots, h'_{i,a_i})).$$

As a consequence, we have $\operatorname{codim}((h'_{i,1}, \dots, h'_{i,a_i})) = a_i$. By Lemma 4.11, we deduce that $\overline{(h_{i,1}, \dots, h_{i,a_i})} = (\mathbf{x}_i)^{d_i}$.

As $\Lambda_s = \{m\}$, for all $2 \leq i \leq r-1, i \neq m, 1 \leq j \leq a_i$ we have $T_s^{i,j} = \varnothing$, hence $\operatorname{supp}(h_{i,j}) \cap (\mathbf{x}_1) = \varnothing$. It follows from (E.i) for J_s that

$$\operatorname{supp}(h_{i,j}) = \operatorname{supp}(h_{i,j}) \setminus (\mathbf{x}_1) = \operatorname{supp}(h_{i,j}) \cap k[\mathbf{x}_2, \dots, \mathbf{x}_r] = \operatorname{supp}(f_{i,j}) \cap k[\mathbf{x}_i] \subseteq k[\mathbf{x}_i].$$

Consequently, we have $h_{i,j} = h'_{i,j}$ for all $2 \leq i \leq r-1, i \neq m, 1 \leq j \leq a_i$, so we have $(\mathbf{x}_i)^{d_i} \subseteq \overline{J_s}$ for all $1 \leq i \leq r-1, i \neq m$. By Lemma 3.4, we also have $\mathfrak{m}^{d_r} \subseteq \overline{J_s}$, hence we have

$$\overline{J_s} \supseteq (\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (h_{m,1}, \dots, h_{m,a_m}) + (\mathbf{x}_{m+1})^{d_{m+1}} + \cdots + (\mathbf{x}_r)^{d_r}.$$

On the other hand, we have

$$J_s \subseteq (\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (h_{m,1}, \dots, h_{m,a_m}) + (\mathbf{x}_{m+1})^{d_{m+1}} + \cdots + (\mathbf{x}_r)^{d_r},$$

so we deduce the equality in Equation (15) with $J = J_s$.

(C.ii) If $h_{m,j} = \sum_{b \in \operatorname{supp}(h_{m,j})} \gamma_b x^b$, we set $h'_{m,j} = \sum_{b \in \operatorname{supp}(h_{m,j}) \cap k[\mathbf{x}_m]} \gamma_b x^b$, which clearly satisfies $h'_{m,j} \in k[\mathbf{x}_m]$. By (E.i) for J_s, we have

$$\operatorname{supp}(h_{m,j} - h'_{m,j}) = \operatorname{supp}(h_{m,j}) \setminus k[\mathbf{x}_m] = \operatorname{supp}(h_{m,j}) \setminus k[\mathbf{x}_2, \dots, \mathbf{x}_r] \subseteq (\mathbf{x}_1).$$

By (E.iii) for J_s, we have $\operatorname{supp}(h_{m,j}) \cap (\mathbf{x}_1) \cap \mathfrak{D} = \varnothing$. Lastly, since $T_s^i = \varnothing$ for all $i \neq m$, we have $T_s^m \neq \varnothing$. If $1 \leq j \leq a_m$ such that $T_s^{i,j} \neq \varnothing$, then $h_{m,j} - h'_{m,j} \neq 0$.

(C.iii) By (D.iv) and (E.iv) for $k = 1, \dots, s$, we have

$$c(J_s) \leq c(J_{s-1}) \leq \cdots \leq c(J_0) \leq c(I).$$

□

Lemma 4.14. *Assume the setting of Assumption 4.1. Let $J \subseteq R$ be an ideal satisfying conditions (C.i) and (C.ii) of Lemma 4.8. Then $c(J) > DP(J)$.*

Proof. Set $J_m = (h_{m,1}, \dots, h_{m,a_m})$. Let

$$J' = (\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + J_m + (\mathbf{x}_{m+1})^{d_{m+1}} + \cdots + (\mathbf{x}_r)^{d_r}.$$

By (C.i), we have $\overline{J} = \overline{J'}$, so it suffices to show $c(J') > DP(J')$. We first prove this result in characteristic $p > 0$.

By assumption, $J_m \not\subseteq \mathfrak{D}$. Define

$$\sigma = \max_{1 \leq j \leq a_m, x^b \in \operatorname{supp}(h_{m,j}), x^b \notin (\mathbf{x}_m)^{d_m}} v(x^b),$$

which satisfies $\sigma < 1$ by condition (C.ii). By Theorem 1.2 we have $c(J_m) > \frac{a_m}{d_m}$. Let $f = h_{1,m}^{t_1} \dots h_{m,a_m}^{t_{a_m}}$ be a generator of $(J_m)^{\nu_{J_m}(p^e)}$ such that $f \notin \mathfrak{m}^{[p^e]}$. Write

$$f = \sum_{b \in \operatorname{supp}(f)} \alpha_b x^b, \quad f' := \sum_{b \in \operatorname{supp}(f) : x^b \notin \mathfrak{m}^{[p^e]}} \alpha_b x^b.$$

As $f \equiv f' \mod \mathfrak{m}^{[p^e]}$, we have $f' J_m \subseteq \mathfrak{m}^{[p^e]}$.

Applying Briançon-Skoda and Lemma 4.11 to the ideal $\frac{J_m+(\mathbf{x}_1)}{(\mathbf{x}_1)}$, we have

$$\frac{(\mathbf{x}_m)^{a_m d_m}+(\mathbf{x}_1)}{(\mathbf{x}_1)} \subseteq \frac{J_m+(\mathbf{x}_1)}{(\mathbf{x}_1)}.$$

Let $\mu_1,\dots,\mu_M$ be a minimal set of monomial generators for $(\mathbf{x}_m)^{a_m d_m}$, and let $\widetilde{\mu_1},\dots,\widetilde{\mu_M} \in J_m$ be homogeneous elements such that $\mu_i - \widetilde{\mu_i} \in (\mathbf{x}_1)$ for all $1 \le i \le M$. Let $\succ$ denote the reverse lexicographic order after reversing the order of the variables. By definition of the reverse lexicographic order, we have $x^b \prec x^{b'}$ for any $x^b \in (\mathbf{x}_1), x^{b'} \in k[\mathbf{x}_2,\dots,\mathbf{x}_r]$. In particular, we have $\operatorname{in}_\succ(\widetilde{\mu_i}) = \operatorname{in}_\succ(\mu_i + (\widetilde{\mu_i}-\mu_i)) = \mu_i$. It follows that $(\mathbf{x}_m)^{a_m d_m} \subseteq \operatorname{in}_\succ(J_m)$.

By construction, we have:

- $f \notin \mathfrak{m}^{[p^e]}$;
- $fJ_m \subseteq J_m^{\nu_{J_m}(p^e)+1} \subseteq \mathfrak{m}^{[p^e]}$.

As $f' \equiv f \mod \mathfrak{m}^{[p^e]}$, we have

- $f' \notin \mathfrak{m}^{[p^e]}$;
- $f'J_m \subseteq \mathfrak{m}^{[p^e]} + J_m^{\nu_{J_m}(p^e)+1} \subseteq \mathfrak{m}^{[p^e]}$.

Let $\operatorname{in}_\succ(f') = x^b$. As no element of $\operatorname{supp}(f')$ is in $\mathfrak{m}^{[p^e]}$, we have $x^b \notin \mathfrak{m}^{[p^e]}$. Additionally, we have

$$\operatorname{in}_\succ(f')\operatorname{in}_\succ(J_m) \subseteq \operatorname{in}_\succ(f'J_m) \subseteq \operatorname{in}_\succ(\mathfrak{m}^{[p^e]}) = \mathfrak{m}^{[p^e]}.$$

By Lemma 3.16, we have

$$(\mathfrak{m}^{[p^e]} : J_m) \subseteq (\mathfrak{m}^{[p^e]} : (\mathbf{x}_m)^{a_m d_m}) = \mathfrak{m}^{[p^e]} + (\mathbf{x}_m)^{a_m(p^e-1)-a_m d_m+1}.$$

As $x^b \notin \mathfrak{m}^{[p^e]}$, we have

$$\operatorname{ord}_{\mathbf{x}_m}(x^b) \ge a_m(p^e-1) - a_m d_m + 1. \tag{31}$$

For $1 \le j \le a_m$, write $h''_{m,j} := h_{m,j} - h'_{m,j}$. We therefore have

$$f = \prod_{i=1}^{a_m}(h'_{m,j}+h''_{m,j})^{t_j} = \prod_{i=1}^{a_m} \sum_{\substack{0\le t'_j,t''_j \le t_j \\ t'_j+t''_j=t_j}} \binom{t_j}{t'_j}(h'_{m,j})^{t_j}(h''_{m,j})^{t''_j}.$$

As $\operatorname{supp}(f') \subseteq \operatorname{supp}(f)$, there exist $(t'_1,t''_1),\dots,(t'_{a_m},t''_{a_m})$ such that $t'_j+t''_j = t_j$ for all $1 \le j \le a_m$ and

$$x^b \in \operatorname{supp}\left((h'_{m,1})^{t'_1}(h''_{m,1})^{t''_1}\dots(h'_{m,a_m})^{t'_{a_m}}(h''_{m,a_m})^{t''_{a_m}}\right).$$

Set $I = (h'_{m,1},\dots,h'_{m,a_m})$. As I is extended from $k[\mathbf{x}_m]$, by Lemma 4.11 we have $I \subseteq (\mathbf{x}_m)^{d_m}$, so

$$t'_1+\dots+t'_{a_m} \le \nu_I(p^e) \le \left\lfloor \frac{a_m(p^e-1)}{d_m} \right\rfloor.$$

Consequently, we have

$$v(x^b) = \sum_{j=1}^{a_m} t'_j v(h'_{m,j}) + \sum_{j=1}^{a_m} t''_j \max_{x^c\in\operatorname{supp}(h''_{m,j})} v(x^b) \tag{32}$$

$$= \sum_{j=1}^{a_m} t'_j + \sum_{j=1}^{a_m} t''_j \max_{x^c\in\operatorname{supp}(h''_{m,j})} v(x^b) \le \sum_{j=1}^{a_m} t'_j + \sigma\sum_{j=1}^{a_m} t''_j \tag{33}$$

$$\le \left\lfloor \frac{a_m(p^e-1)}{d_m} \right\rfloor + \sigma\left(\nu_{J_m}(p^e) - \left\lfloor \frac{a_m(p^e-1)}{d_m} \right\rfloor\right). \tag{34}$$

As in Lemma 3.16, let $\mathfrak{a}_\lambda, \mathfrak{a}_\lambda^+$ denote the ideals $\{f \in R : v(f) \geq \lambda\}, \{f \in R : v(f) > \lambda\}$ respectively. Let t_e denote the quantity in Equation (34) and set $u_e := (p^e - 1)(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r})$. It follows from Lemma 3.16 that

$$f' \notin \mathfrak{m}^{[p^e]} + \mathfrak{a}_{t_e}^+ = (\mathfrak{m}^{[p^e]} : \mathfrak{a}_{u_e - t_e}).$$

Let $x^{b'} \in \mathfrak{a}_{u_e-t_e}$ such that $x^{b+b'} \notin \mathfrak{m}^{[p^e]}$. Write $x^{b'} = x^{b''}y$ where $y \in k[\mathbf{x}_m]$ and $x^{b''} \in k[\mathbf{x}_1, \ldots, \widehat{\mathbf{x}_m}, \ldots, \mathbf{x}_r]$. As $yx^b \notin \mathfrak{m}^{[p^e]}$, by Equation (31) we have

$$\operatorname{ord}_{(\mathbf{x}_m)}(y) \leq (p^e - 1)a_m - \operatorname{ord}_{(\mathbf{x}_m)}(x^b) \leq a_m d_m,$$

hence $v(x^{b''}) = v(x^{b'}) - v(y) \geq u_e - t_e - a_m$. As $\mathfrak{D} = \overline{(\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_r)^{d_r}}$, by Briançon-Skoda we have

$$x^{b''} \in \mathfrak{D}^{\lfloor u_e - t_e \rfloor - a_m} \subseteq ((\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - a_m - n}.$$

Since $x^{b''} \notin (\mathbf{x}_m)$, we in fact have

$$x^{b''} \in ((\mathbf{x}_1)^{d_1} + \cdots + (\mathbf{x}_{m-1})^{d_{m-1}} + (\mathbf{x}_{m+1})^{d_{m+1}}(\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - a_m - n} \subseteq (J')^{\lfloor u_e - t_e \rfloor - a_m - n}.$$

It follows that $\nu_{J'}(p^e) \geq \nu_{J_m}(p^e) + \lfloor u_e - t_e \rfloor - a_m - n$. Dividing by p^e and letting $e \to \infty$, we obtain

$$\begin{aligned} c(J') &\geq c(J_m) + \lim_{e\to\infty} \frac{u_e}{p^e} - \lim_{e\to\infty} \frac{t_e}{p^e} \\ &= c(J_m) + \left(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r}\right) - \left(\frac{a_m}{d_m}(1-\sigma) + \sigma c(J_m)\right) \\ &= (1-\sigma)\left(c(J_m) - \frac{a_m}{d_m}\right) + \left(\frac{a_1}{d_1} + \cdots + \frac{a_r}{d_r}\right) \end{aligned}$$

Since $\sigma < 1$ and $c(J_m) > \frac{a_m}{d_m}$, it follows that the above quantity exceeds $DP(J')$.

In characteristic zero, one notes that for any ideal J satisfying conditions (i)-(iv), the reduction of the pair (R, J) to characteristic $p \gg 0$ satisfies conditions (i)-(iv). Moreover, the quantity σ is constant for $p \gg 0$. Assuming the reduction notation of Definition 3.14, we have

$$c(J) = \lim_{\substack{\mu \in \operatorname{Spec} A \\ |A/\mu| \to \infty}} c(J_\mu) \geq (1-\sigma) \lim_{\substack{\mu \in \operatorname{Spec} A \\ |A/\mu| \to \infty}} c(J_{m,\mu}) + DP(J) = (1-\sigma)c(J_m) + DP(J) > DP(J).$$

□

Lemmas 4.6, 4.8 and 4.14 combine to give us a proof of Theorem 4.22 in the case of a complete intersection.

Proposition 4.15. *Assume the setup of Assumption 4.1 and suppose $c(I) = DP(I)$. Then there exists $\varphi \in \mathrm{GL}_n(k)$ depending only on $I_1, \ldots, I_{r-1}$ such that $\overline{\varphi^* I} = \mathfrak{D}$.*

Proof. Using Lemma 4.6, we produce $\varphi \in \mathrm{GL}_n(k)$ such that $\varphi^* I$ satisfies Equation (10). By Lemmas 4.8 and 4.14, we have $\overline{\varphi^* I} = \mathfrak{D}$. □

4.4. **Generalizations.**

Lemma 4.16. *Let $R = k[x_1, \ldots, x_n]$ and let $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $I \subseteq R$ be a homogeneous ideal and $J \subseteq \mathfrak{m}$ any ideal. Then we have*

$$\bigcap_{m>0} \overline{I + J^m} = \overline{I}.$$

Proof. By [17, Corollary 6.8.5], we have

$$\overline{IR_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + \mathfrak{m}^m R_\mathfrak{m}} = \overline{IR_\mathfrak{m}}.$$

As $\overline{I} = \overline{I}R_\mathfrak{m} = \overline{IR_\mathfrak{m}} \cap R$, we have the following, from which the claim follows.

$$\overline{I} \subseteq \bigcap_{m>0} \left(\overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \cap R\right) \subseteq \overline{IR_\mathfrak{m}} \cap R = \overline{I}.$$

□

This allows us to prove a version of Theorem 4.22 for complete intersections of smaller codimension.

Proposition 4.17. *Let k be an algebraically-closed field and set $R = k[x_1, \dots, x_n]$. Set $\mathfrak{m} = (x_1, \dots, x_n)$. Suppose $r < n$ and $I = (f_1, \dots, f_r)$ is a complete interesection, where f_i is homogeneous of degree d_i and $d_1 \le \dots \le d_r$. Then $c(I) \ge \frac{1}{d_1} + \dots + \frac{1}{d_r}$ with equality if and only if there exist coordinates for R such that*

$$\overline{I} = \overline{(x_1^{d_1}, \dots, x_r^{d_r})}.$$

Proof. Let $\mathfrak{D} = \overline{(x_1^{d_1}, \dots, x_r^{d_r})}$. For an ideal J and an integer d, we let $[J]_{\le d}$ denote the vector space of homogeneous forms in J of degree $\le d$. Recall that the total space of forms of degree $\le d$ in R is finite-dimensional. It follows that the infinite descending chain $[\overline{\mathfrak{D} + \mathfrak{m}}]_{\le d_r} \supseteq [\overline{\mathfrak{D} + \mathfrak{m}^2}]_{\le d_r} \supseteq \dots$, the limit of which is $[\mathfrak{D}]_{\le d_r}$ by Lemma 4.16, must eventually stabilize. Let $e_0 \gg d_r$ such that $[\overline{\mathfrak{D} + \mathfrak{m}^e}]_{\le d_r} = [\mathfrak{D}]_{d_r}$ for all $e \ge e_0$.

Let $\ell_{r+1}, \dots, \ell_n$ be general linear forms such that $(f_1, \dots, f_r, \ell_{r+1}, \dots, \ell_r)$ is a complete intersection. Choose $e > \max(d_r, e_0)$ such that $[\overline{I + (\ell_{r+1}, \dots, \ell_n)^e}]_{\le d_r} = [\overline{I}]_{\le d_r}$.

Let $J_e := (\ell_{r+1}^e, \dots, \ell_n^e)$. Then $I + J_e$ is a complete intersection of type $(d_1, \dots, d_r, e, \dots, e)$. By the general bound $c(\mathfrak{a} + \mathfrak{b}) \le c(\mathfrak{a}) + c(\mathfrak{b})$, we have $c(I + J_e) \le c(I) + \frac{n-r}{e}$. On the other hand, by Corollary 3.18, if L is a general linear space of codimension $n-r$, then $c(I+J_e) \ge c((I+J_e)|_L) + \frac{n-r}{e}$. By Lemma 3.10, we have $\overline{I|_L} = \overline{(I + J_e)|_L}$. It follows that

$$c(I + J_e) \ge c((I + J_e)|_L) + \frac{n-r}{e} = c(I|_L) + \frac{n-r}{e} \ge DP(I|_L) + \frac{n-r}{e} = c(I) + \frac{n-r}{e}.$$

We deduce that $c(I + J_e) = \frac{1}{d_1} + \dots + \frac{1}{d_r} + \frac{n-r}{e} = DP(I + J_e)$. By Proposition 4.15, there exist coordinates for R such that $\overline{I + J_e} = \overline{\mathfrak{D} + \mathfrak{m}^e}$. As $\overline{J_e} = \overline{(\ell_{r+1}, \dots, \ell_n)^e}$, we have

$$[\overline{I}]_{\le d_r} = [\overline{I + J_e}]_{\le d_r} = [\overline{\mathfrak{D} + \mathfrak{m}^e}]_{\le d_r} = [\mathfrak{D}]_{\le d_r}.$$

The generators of I are contained in $\mathfrak{D}$ and the generators of $\mathfrak{D}$ are contained in $\overline{I}$, so $\overline{I} = \mathfrak{D}$. □

Lemma 4.18. *Let k be an uncountably infinite field. Let $R = k[x_1, \dots, x_n]$ and set $\mathfrak{m} = (x_1, \dots, x_n)$. Suppose $I \subseteq R$ is a homogeneous ideal. As in Lemma 3.1, for $1 \le j \le n$, let $\pi_j : R \to R/(x_{j+1}, \dots, x_n) \cong k[x_1, \dots, x_j]$ denote the projection map and $\iota_j : k[x_1, \dots, x_j] \to k[x_1, \dots, x_n]$ the usual embedding. Let $>$ denote the reverse lexicographic order.*

Let $\varphi \in \mathrm{GL}_n(k)$ be very general: for now, we impose the condition that for all $m > 0$, we have $\mathrm{in}_>(\varphi^* I^m) = \mathrm{gin}_>(I^m)$; we will impose countably many additional conditions in Lemma 4.21. For $1 \le j \le n, m > 0$, set $\mathfrak{a}_{j,m} := \mathrm{in}_>(\pi_j(\varphi^* I^m))$. For $j > 0, 1 \le i \le j$, let $b_{i,j}$ denote the ith unit vector of $\mathbb{R}^j$. Set $p_j(i) := \inf\{t : tb_{i,j} \in \Gamma(\mathfrak{a}_{j,\bullet})\}$. Then for all j, we have $p_j(j) = p_n(j)$.

Proof. By Lemma 3.1, we have $\iota_j(\mathfrak{a}_{j,\bullet}) \subseteq \mathfrak{a}_{n,\bullet}$ for all $1 \le j \le n$, so we have $p_n(j) \le p_j(j)$. For the reverse direction, set $t = p_n(j)$. Since $tb_{j,n} \in \overline{\bigcup_{m>0} \frac{1}{2^m}\Gamma(\mathfrak{a}_{n,2^m})}$, there exists a sequence $\{a_m = (a_{m,1}, \dots, a_{m,n})\}_{m>0}$ such that $a_m \in \Gamma(\mathfrak{a}_{n,2^m})$ for all m and $\lim_{m\to\infty} 2^{-m} a_m = tb_{j,n}$. For any choice of $\{(a_{m,1}, \dots, a_{m,n})\}_{m>0}$, we also have $(\lceil a_{m,1}\rceil, \dots, \lceil a_{m,n}\rceil) \in \Gamma(\mathfrak{a}_{n,2^m})$

and $\lim_{m\to\infty} \frac{(\lceil a_{m,1}\rceil,\ldots,\lceil a_{m,n}\rceil)}{2^m} = tb_{j,n}$. We may therefore assume without loss of generality that $a_m \in (\mathbb{Z}^+)^n$ for all $m > 0, 1 \le i \le n$, hence for all $m > 0$, we have $x^{a_m} \in \overline{\mathfrak{a}_{n,2^m}}$.

By [12, Theorem 2.1], $\overline{\mathfrak{a}_{n,2^m}}$ is Borel-fixed, so we have $x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j}+\cdots+a_{m,n}} \in \overline{\mathfrak{a}_{n,2^m}}$. Further note that $\mathfrak{a}_{j,m} = \pi_j(\mathfrak{a}_{n,m})$ by Lemma 3.1. By Proposition 2.36(iii), we conclude

$$\pi_j(x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j}+\cdots+a_{m,n}}) \in \pi_j(\overline{\mathfrak{a}_{n,2^m}}) \subseteq \overline{\pi_j(\mathfrak{a}_{n,2^m})} = \overline{\mathfrak{a}_{j,2^m}}. \tag{35}$$

It follows that

$$tb_{j,j} = \lim_{m\to\infty} \frac{(a_{m,1},\ldots,a_{j-1},a_j+\cdots+a_n)}{m} \in \Gamma(\mathfrak{a}_{j,\bullet}),$$

which proves $p_j(j) \le t = p_n(j)$. ☐

Lemma 4.19. *Assume the setup of Lemma 4.18. There exists a sequence $\{a'_m\}_{m>0}$ such that for all $m > 0$, we have $x^{a'_m} \in \mathfrak{a}_{j,2^m}$ and $\lim_{m\to\infty} 2^{-m}a'_m = p_j(j)b_{j,j}$.*

Proof. Consequently, there exists a sequence $\{a_m = (a_{m,1},\ldots,a_{m,j})\}_{m>0}$ such that for all $m > 0$ we have $\lim_{m\to\infty} 2^{-m}a_m = tb_{j,j}$ and $a_m \in \Gamma(\mathfrak{a}_{2^m})$.

First, note that $\Gamma(\mathfrak{a}_{2^m}) = \operatorname{conv}(\log(x^u) : x^u \in \mathfrak{a}_{2^m})$. If we triangulate $\Gamma(\mathfrak{a}_{2^m})$, we may choose $\{u_{m,i} = (u_{m,i,1},\ldots,u_{m,i,j})\}_{i=0}^{j}$ such that $x^{u_{m,i}} \in \mathfrak{a}_{2^m}$ and $a_m \in \operatorname{conv}(u_{m,0},\ldots,u_{m,j+1})$. Reorder the $u_{m,i}$ so that $u_{m,0,j} \le \cdots \le u_{m,j,j}$. Since $a_{m,j}$ is the average of the $u_{m,i,j}$, we have $u_{m,0,j} \le a_{m,j}$. For $i < j$, we similarly have $\frac{u_{m,0,i}}{j+1} \le \frac{u_{m,0,i}+\cdots+u_{m,j,i}}{j+1} = a_{0,i}$, so $u_{m,0,i} \le (j+1)a_{0,i}$.

For all $m > 0$, set $a'_m = u_{m,0}$. Then we have $\lim_{m\to\infty} a'_{m,i} \le (j+1)\lim_{m\to\infty} a_{m,i} = 0$ for all $i < j$, and

$$t \le \liminf_{m\to\infty} 2^{-m}a'_{m,j} \le \lim_{m\to\infty} 2^{-m}a_{m,j} = t.$$

It follows that $\lim_{m\to\infty} 2^{-m}a'_m = tb_{j,j}$ and for all $m > 0, x^{a'_m} \in \mathfrak{a}_{2^m}$. ☐

Lemma 4.20. *Let k be an algebraically-closed field and $R = k[x_1,\ldots,x_j]$. Let $\mathfrak{q}$ be a homogeneous prime ideal of codimension $j-1$ with $x_j \notin \mathfrak{q}$. If $>$ denotes the reverse lexicographic order, then for all $m > 0$ we have $\operatorname{in}_>(\mathfrak{q}^m) = (x_1,\ldots,x_{j-1})^m$.*

Proof. Since k is algebraically-closed, there exist linear forms $\ell_1,\ldots,\ell_{j-1} \in R_1$ such that $q = (\ell_1,\ldots,\ell_{j-1})$. It follows that $[\operatorname{in}_>(\mathfrak{q})]_1 = \operatorname{in}_>(\ell_1 \wedge \cdots \wedge \ell_{j-1})$, which is equal to $x_1 \wedge \cdots \wedge x_{j-1}$ by the fact that $x_j \notin \operatorname{span}(\ell_1,\ldots,\ell_{j-1})$. Consequently, we have $(x_1,\ldots,x_j) \subseteq \operatorname{in}_>(\mathfrak{q})$. By [8, Theorem 15.17], $\operatorname{in}_>(\mathfrak{q})$ and $\mathfrak{q}$ have the same Hilbert series, so we in fact have $(x_1,\ldots,x_j) = \operatorname{in}_>(\mathfrak{q})$.

For $m > 1$, a similar analysis applies. We have the standard containment $(x_1,\ldots,x_{j-1})^m = \operatorname{in}_>(\mathfrak{q})^m \subseteq \operatorname{in}_>(\mathfrak{q}^m)$. As $(x_1,\ldots,x_{j-1})^m$ has the same Hilbert series as $\mathfrak{q}^m$, the result follows. ☐

Lemma 4.21. *Assume the setup of Lemmas 4.18 and 4.19. Further assume that $k = \overline{k}$. Write $I = I_1 + \cdots + I_r$, where each I_i is generated by d_i-forms and $d_1 < \cdots < d_r$. For $1 \le j \le r$, set $h_i := \operatorname{codim}(I_1 + \cdots + I_i) - \operatorname{codim}(I_1 + \cdots + I_{i-1})$. For $1 \le i \le n$, we also define $q_i := j$, where $1 \le j \le r$ such that $h_1 + \cdots + h_{j-1} < i \le h_1 + \cdots + h_j$. Then for all $1 \le j \le n$, we have $p_n(j) = d_{q_j}$.*

Proof. Before we begin the proof, we first state the additional generality conditions on φ. For all $m > 0$, assume that $\operatorname{in}_>(\pi_j(\varphi^* I^m)) = \operatorname{gin}_>(\pi_j(\varphi^* I^m)) = \pi_j(\operatorname{in}_>(\varphi^* I^m))$; this is possible by repeated application of [3, Theorem 1.13.]. Since $\operatorname{codim} \pi_j(\varphi^*(I_1 + \cdots + I_{q_j - 1})) < j$, we may also choose φ such that $x_j \notin \sqrt{\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))}$. Each of these conditions is satisfied by a general choice of φ, so they may be realized simultaneously.

Set $J = \pi_j(\varphi^* I)$. By construction of φ, in the language of Lemma 4.18 we have $\mathfrak{a}_{j,m} = \operatorname{in}_>(J^m)$. By construction of φ, we have $x_j \notin \sqrt{\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))}$, so we may choose a minimal prime $\mathfrak{p}$ over $\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))$ such that $x_j \notin \mathfrak{p}$. As $\operatorname{codim} \mathfrak{p} \le j-1$, we may choose a homogeneous prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$ such that $\operatorname{codim} \mathfrak{q} = j-1$ and $x_j \notin \mathfrak{q}$. By Lemma 4.20, we have $\operatorname{in}_>(\mathfrak{q}^m) = (x_1,\ldots,x_{j-1})^m$ for all $m > 0$.

By Lemma 4.19, choose a sequence $\{a_m\}_{m>0}$ such that $x^{a_m} \in \mathfrak{a}_{2^m}$ for all $m > 0$ and $\lim_{m\to\infty} 2^{-m} a_m = p_j(j) b_{j,j}$. Let $e_m := a_{m,1} + \cdots + a_{m,j}$. For all $m > 0$, we have

$$[J^{2^m}]_{e_m} = \left[\sum_{\substack{\gamma_1+\cdots+\gamma_r=2^m \\ \gamma_1 d_1+\cdots+\gamma_r d_r \le e_m}} \pi_j(\varphi^* I_1)^{\gamma_1} \cdots \pi_j(\varphi^* I_r)^{\gamma_r} \right]_{e_m}$$

For $1 \le i \le q_j - 1$, we have $I_i \subseteq \mathfrak{q}$. For $q_j \le i \le r$, we have $I_i \subseteq (x_1, \ldots, x_j) = \mathfrak{m}$. It follows that

$$[J^{2^m}]_{e_m} \subseteq \left[\sum_{\substack{\gamma_1+\cdots+\gamma_r=2^m \\ \gamma_1 d_1+\cdots+\gamma_r d_r \le e_m}} \mathfrak{q}^{\gamma_1+\cdots+\gamma_{q_j-1}} \mathfrak{m}^{\gamma_{q_j}+\cdots+\gamma_r} \right]_{e_m} \subseteq \left[\sum_{\substack{\alpha+\beta=2^m \\ \beta d_{q_j} \le e_m}} \mathfrak{q}^\alpha \mathfrak{m}^\beta \right]_{e_m} \subseteq \left[\mathfrak{q}^{2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor} \right]_{e_m}.$$

Taking initial ideals of both sides, we have $x^{a_m} \in (x_1, \ldots, x_j)^{2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor}$. Consequently, we have $e_m - a_{m,j} = a_{m,1} + \cdots + a_{m,j-1} \ge 2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor$. As $\lim_{m\to\infty} 2^{-m}(a_{m,1} + \cdots + a_{m,j-1}) = 0$, this yields

$$0 \le \liminf_{m\to\infty} 2^{-m} \left(2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor \right) = 1 - \frac{1}{d_{q_j}} \liminf_{m\to\infty} \frac{a_{m,j}}{d_{q,j}} = 1 - \frac{p_j(j)}{d_{q_j}}.$$

From the above equation, we have $p_j(j) \ge d_{q_j}$. For the reverse containment, we have by Lemma 3.4 that $\mathfrak{m}^{d_{q_j}} \subseteq \overline{J}$. It follows that $x_j^{(m+j-1)d_j} \in J^m$ for all $m > 0$, hence $p_j(j) \le d_{q_j}$. □

We are now able to prove Theorem 4.22.

Theorem 4.22. *Let k be an algebraically-closed field. Let $R = k[x_1, \ldots, x_n]$ and let $I \subseteq R$ be a $\mathfrak{m}$-primary homogeneous ideal. If $DP(I) = c(I)$, then there exist integers $d_1, \ldots, d_n$ and $\varphi \in \mathrm{GL}_n(k)$ such that*

$$\varphi^* \overline{I} = \overline{\left(x_1^{d_1}, \ldots, x_n^{d_n} \right)}.y$$

Proof. By Corollary 3.12, we have

$$\overline{\mathbb{R} \setminus \Gamma_>(\mathfrak{a}_\bullet)} = \operatorname{conv} \left(\vec{0}, (e_1(I), 0, \ldots, 0), \left(0, \frac{e_2(I)}{e_1(I)}, 0, \ldots, 0 \right), \ldots, \left(0, \ldots, 0, \frac{e_n(I)}{e_{n-1(I)}} \right) \right).$$

Assume the h_i, q_i notation from Lemma 4.21. If we let $L := \overline{k((t))}$, then L is uncountably infinite and algebraically closed. The generic initial ideal is stable under field extension, so applying Lemma 4.21 to $I \otimes_k L$, we have $\frac{e_j(I)}{e_{j-1}(I)} = d_{q_j}$ for all $1 \le j \le n$.

Let $J \subseteq I$ be an ideal generated by h_i general elements of I_i for each $1 \le i \le r$. Then J is a homogeneous $(d_{q_1}, \ldots, d_{q_n})$-complete intersection, so by Lemma 3.11, we have $DP(J) = DP(I)$. It follows from Proposition 2.19 that $\overline{J} = \overline{I}$, so we have $c(J) = c(I) = DP(I) = DP(J)$. By Proposition 4.15, there exists $\varphi \in \mathrm{GL}_n(k)$ such that

$$\varphi^* \overline{I} = \varphi^* \overline{J} = \overline{\left(x_1^{d_{q_1}}, \ldots, x_n^{d_{q_n}} \right)}.$$

□

5. Future Work

In [4], Bivià-Ausina poses the question as to whether or not Theorem 4.22 holds for arbitrary ideals of $\mathcal{O}_n$ with $c = DP$. We answer this question in the negative.

Example 5.1. Let $R = \mathcal{O}_2$. Let x, y be local coordinates for $\mathcal{O}_2$ and let $I = (x + y^2, y^3) \subseteq \mathcal{O}_2$. Then $c(I) = \frac{4}{3} = DP(I)$. As $\overline{I}$ is not homogeneous, the same is true for $\varphi^* I$ for any linear change of coordinates $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$, so $\varphi^* \overline{I}$ cannot be a diagonal monomial ideal. If we instead allow ourselves to consider *holomorphic* changes of coordinates, however, there is still hope. Let U denote a small neighborhood of $0 \in \mathbb{C}^2$ and let $\varphi : U \to \mathbb{C}^2$ such that $\varphi(x + y^2) = x, \varphi(y) = y$. Then $\varphi^* I = (x, y^3)$ is a diagonal monomial ideal as expected.

With the above example in mind, we believe Theorem 4.22 is evidence for a stronger conjecture in the local setting.

Conjecture 5.2. *Let k be an algebraically-closed field and $(R, \mathfrak{m}) = (k[\![x_1, \ldots, x_n]\!], (\underline{x}))$. Let $I \subseteq R$ be $\mathfrak{m}$-primary with $c(I) = DP(I)$. Then $e_{j+1}(I)/e_j(I) \in \mathbb{Z}^+$ for $0 \leq j \leq n - 1$. Moreover, there exists an automorphism $\varphi : R \to R$ with*

$$\varphi(\overline{I}) = \overline{(x_1^{e_1(I)}, x_2^{e_2(I)/e_1(I)}, \ldots, x_n^{e_n(I)/e_{n-1}(I)})}.$$

Lastly, we pose a question in the analytic setting.

Question 5.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, hyperconvex domain containing 0. Let $\phi : \Omega \to \mathbb{R} \cup \{-\infty\}$ be plurisubharmonic with an isolated singularity at 0. Suppose $c(\phi) = DP(\phi)$ (see Section 2.3 for relevant definitions). Must there exist $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, biholomorphic at 0, such that $\varphi(0) = 0$ and*

$$(\phi \circ \varphi)(z) = \left(\log \max_{0 \leq i \leq n-1} \frac{e_{i+1}(\phi)|z_i|}{e_i(\phi)} \right) + O(1)?$$

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