CLASSIFICATION OF MINIMAL SINGULARITY THRESHOLDS

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ABSTRACT. Let k be a field of characteristic zero, $R = k[x_1, \dots, x_n]$, and $I \subseteq R$ an ideal primary to $(x_1, \dots, x_n)$. By a 2014 result of Demailly and Pham, we have $\operatorname{lct}(I) \geq \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)}$, where $\operatorname{lct}(I)$ is the log canonical threshold of $(\operatorname{Spec} R, \operatorname{Spec} R/I)$ and $e_j(I)$ is the jth Segre number of I.

If instead $\operatorname{char} k = p > 0$, we show that the F-pure threshold of (R, I) satisfies the same lower bound. In both characteristic zero and positive characteristic, we classify all homogeneous ideals which attain the lower bound.

1. INTRODUCTION

We consider the log canonical threshold (lct) and F-pure threshold (fpt) of a pair (X, Y) where X is a smooth K-scheme and Y a subscheme supported at a point. The lct in characteristic zero and the fpt in positive characteristic have attracted considerable attention in algebraic geometry due to their connections with the Minimal Model Program and singularity theory. In recent years, many authors [10, 5, 7, 4, 9, 17] have proven results comparing the lct to multiplicity-like invariants of the pair (X, Y).

In this paper, we will consider a lower bound on the lct due to Demailly and Pham [7] in terms of the Segre numbers of Y (Theorem 1.2). We'll show that in positive characteristic, the analogous bound holds for fpt. Our main contribution (Theorem 4.14) is to classify the homogeneous pairs (X, Y) for which the lct or fpt equals the lower bound.

Theorem 1.1 ([5], Theorem 0.1)**.** *Let $(R, \mathfrak{m})$ be a regular local ring essentially of finite type over an algebraically-closed field of characteristic zero. Suppose I is $\mathfrak{m}$-primary. Let $e(I)$ denote the Hilbert multiplicity of I and $n = \dim R$. Then we have*

$$e(I) \geq \left(\frac{n}{\operatorname{lct}(I)}\right)^n$$

with equality if and only if the integral closure $\overline{I}$ of I is a power of $\mathfrak{m}$.

A variant of Theorem 1.1 due to Demailly and Pham uses the Segre numbers of I to obtain a stronger lower bound on $\operatorname{lct}(I)$.

Theorem 1.2 ([7], Theorem 1.2)**.** *Let $(\mathcal{O}_n, \mathfrak{m})$ denote the ring of germs at zero of holomorphic functions $\mathbb{C}^n \to \mathbb{C}$. Let I be an $\mathfrak{m}$-primary ideal and let $e_j(I)$ denote the jth Segre number of I (see Section 2.3). Then we have*

$$\frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} \leq \operatorname{lct}(I). \tag{1}$$

Moreover, this bound is attained by the ideal $I = (x_1^{d_1}, \dots, x_n^{d_n})$ for any $d_1, \dots, d_n \in \mathbb{Z}^+$.

We will refer to the left-hand side of Equation (1) as the Demailly-Pham invariant of I, denoted $DP(I)$ (see Section 2.3). In this paper, we classify homogeneous ideals I that achieve equality in Equation (1).

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Theorem 4.14. *Let K be an algebraically-closed field of characteristic zero. Let $R = K[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and let $I \subseteq R$ be a $\mathfrak{m}$-primary homogeneous ideal. If $DP(I) = \operatorname{lct}(I)$, then there exist integers $d_1, \dots, d_n$ such that, in suitable coordinates, we have*

$$\overline{I} = \overline{\left(x_1^{d_1}, \dots, x_n^{d_n}\right)}.$$

If instead char $K = p > 0$*, then the same result holds with* $\operatorname{lct}(I)$ *replaced by* $\operatorname{fpt}(I)$*.*

We will briefly outline the proof of the theorem. Assume I is an ideal satisfying equality in Equation (1). Write $I = I_1 + \cdots + I_r$, where I_j is generated by d_j-forms.

(1) Using results from [4], we control the generic initial ideals $\{\operatorname{gin}(I^n)\}_{n \geq 1}$.
(2) Using (1), we obtain a formula for $e_j(I)$ in terms of the numbers $d_j, \operatorname{codim}(I_1 + \cdots + I_j)$, which allows us to reduce to the case of a complete intersection.
(3) We prove the result by induction on the number of distinct degrees $d_1, \dots, d_r$.

In the case $r = 1$, any $\mathfrak{m}$-primary ideal I generated by d-forms automatically satisfies $\overline{I} = \mathfrak{m}^d$, so there is no way to use $r = 1$ as a useful base case for our induction. Instead, we use $r = 2$. In this case, we show (Lemma 3.18)that $c(I) = DP(I)$ if and only if $c(I_1) = \operatorname{codim}(I_1)/d_1$. As $\overline{I} = \overline{\overline{I_1} + \mathfrak{m}^{d_2}}$, it suffices to show that $\overline{I_1} = (x_1, \dots, x_{\operatorname{codim}(I)})^{d_1}$ in suitable coordinates. In characteristic zero, this follows from [10, Theorem 3.5]. In positive characteristic, this fact is recorded below.

Theorem 1.3 ([1], Theorem 3.17)**.** *Let K be a field of characteristic $p > 0$. Let I be a homogeneous ideal in $K[x_1, \dots, x_n]$ generated by polynomials of degree d and set $h = \operatorname{codim}(I)$. Suppose that K is algebraically-closed. Then $\operatorname{fpt}(I) = h/d$ if and only if $\overline{I} = (x_1, \dots, x_h)^d$ up to change of coordinates.*

2. Preliminaries

2.1. **F-Pure and Log Canonical Thresholds.** We begin with a formal definition of the log canonical threshold. For a detailed introduction, see [18].

Definition 2.1 (Log Resolution)**.** Let X be a smooth variety over a characteristic zero field with $Y \subseteq X$ a proper closed subvariety with defining ideal $\mathfrak{a}$. Let W be a smooth variety. A projective morphism $\pi : W \to X$ is a *log resolution* of (X, Y) if π is an isomorphism over $X \setminus Y$ and the inverse image $\mathfrak{a} \cdot \mathcal{O}_W$ is the ideal of a Cartier divisor D such that $D + K_{W/X}$ has simple normal crossings.

The following result gives a concise definition of the log canonical threshold.

Definition 2.2 (Log Canonical Threshold, [18] Theorem 1.1)**.** Let X be a smooth variety with $Y \subseteq X$ a closed subvariety with defining ideal $\mathfrak{a}$. By Hironaka's theorem on resolution of singularities in characteristic zero, there exists a log resolution $\pi : W \to X$ of the pair (X, Y). If $E_1, \dots, E_N$ are the exceptional divisors of π, then we can write

$$D = \sum_{i=1}^{N} a_i E_i \qquad \text{and} \qquad K_{W/X} = \sum_{i=1}^{N} k_i E_i.$$

The quantity $\min_i \frac{k_i+1}{a_i}$ does not depend on π and is called the **log canonical threshold** of (X, Y).

Definition 2.3 (F-Pure Threshold, [20] Chapter 4.4)**.** Let R be an F-finite ring, $\mathfrak{a} \subseteq R$ an ideal, and $t \in \mathbb{R}^+$. The pair $(R, \mathfrak{a}^t)$ is *sharply F-split* if for infinitely many $e > 0$, the map

$$\mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \operatorname{Hom}(F^e_* R, R) \to R$$

is surjective. The *F-pure threshold* of the pair $(R, \mathfrak{a})$ is the supremum of all t such that $(R, \mathfrak{a}^t)$ is sharply F-split.

In practice, we will not use the above two definitions. Instead, we use the following two propositions characterizing lct and fpt respectively.

Proposition 2.4 ([20], following Definition 4.29)**.** *Let A be a finite-type $\mathbb{Z}$-algebra and $\mathfrak{a} \subseteq A[x_1, \ldots, x_n]$ an ideal. Set $K = Frac(A)$. Then we have*

$$\operatorname{lct}(K[x_1, \ldots, x_n], \mathfrak{a} \otimes_A K) = \lim_{\mu \in \max \operatorname{Spec} A, |A/\mu| \to \infty} \operatorname{fpt}(A/\mu[x_1, \ldots, x_n], \mathfrak{a} \otimes_A A/\mu).$$

Proposition 2.5 ([20], Exercises 4.19-4.20)**.** *Let $(R, \mathfrak{m})$ be an F-finite regular local ring. Then the F-pure threshold of the pair (R, I^t) is equal to*

$$\sup \left\{ \frac{\nu}{p^e} : I^\nu \notin \mathfrak{m}^{[p^e]} \right\}.$$

In fact, let $\nu_I(p^e) = \max\{r : I^r \notin \mathfrak{m}^{[p^e]}\}$. Then the F-pure threshold of $(R, \mathfrak{a})$ is equal to the limit $\lim_{e\to\infty} \nu_I(p^e)/p^e$.

If instead R is a polynomial ring over an F-finite field and $I \subseteq R$ a homogeneous ideal, then the same results hold when we let $\mathfrak{m}$ denote the homogeneous maximal ideal of R.

Many of the results we will make sense for both F-pure and log canonical thresholds, so we will introduce the following notation to avoid stating the same results once each for characteristic zero and positive characteristic.

Definition 2.6 (Notation: Nonspecified Threshold)**.** Let $R = K[x_1, \ldots, x_n]$ and $I \subseteq R$ a homogeneous ideal. We define the quantity $c(R, I)$ as follows:

$$c(R, I) = \begin{cases} \operatorname{fpt}(R, I) & \text{char } R = p > 0 \\ \operatorname{lct}(R, I) & \text{char } R = 0 \end{cases}.$$

If the context is clear, we will use $c(I)$ for short.

We will require the following essential fact:

Proposition 2.7. *Let $R = K[x_1, \ldots, x_n]$. Let $>$ be a monomial order. Let $I \subseteq R$ be an ideal, and $\operatorname{in}_>(I)$ the initial ideal of I with respect to $>$. Then $c(\operatorname{in}_>(I)) \leq c(I)$.*

Proof. For characteristic zero, see [6] for the semicontinuity of the lc threshold. For positive characteristic, see [22], the claim preceding Remark 4.6. ☐

2.2. **Newton Polytopes of Monomial Ideals.** When working with monomial ideals, one often identifies a monomial $x_0^{a_0} \cdots x_n^{a_n}$ with the point $(a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$. For future reference, it will help to give a name to this identification.

Definition 2.8. Let K be a field. We define the map

$$\log : \{\text{monomials in } K[x_0, \ldots, x_n]\} \to \mathbb{Z}_{\geq 0}^{n+1} \quad \text{by the rule} \quad \log(x_0^{a_0} \cdots x_n^{a_n}) = (a_0, \ldots, a_n).$$

Definition 2.9. Let $\mathfrak{a} \subseteq K[x_0, \ldots, x_n]$ be a monomial ideal. Then the *Newton Polytope* of I, denoted $\Gamma(\mathfrak{a})$, is the convex hull in $\mathbb{R}^{n+1}$ of $\log(\mathfrak{a})$. Later on, we will let $\operatorname{conv}(-)$ denote the convex hull of a set.

Remark 2.10. We record several properties of $\Gamma(\mathfrak{a})$.

(i) $\Gamma(\mathfrak{a})$ is a closed, convex, unbounded subset of the first orthant of $\mathbb{R}^n$.
(ii) When $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, the complement of $\Gamma(\mathfrak{a})$ inside the first orthant is an open, bounded polyhedron.
(iii) For two ideals $\mathfrak{a}, \mathfrak{b}$, the Minkowski sum of $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{b})$ is equal to $\Gamma(\mathfrak{a}\mathfrak{b})$. In particular, $\Gamma(\mathfrak{a}^n) = n\Gamma(\mathfrak{a})$.

Definition 2.11. Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal and $t \in \mathbb{Z}^+$. We let $[I]_t$ denote the vector space of t-forms in I.

Proposition 2.12. *Let $\mathfrak{a} \subseteq K[x_1, \ldots, x_n]$ be a monomial ideal. Then*

$$c(\mathfrak{a}) = \frac{1}{\mu}, \text{ where } \mu = \inf\{t : t\vec{1} \in \Gamma(\mathfrak{a})\}.$$

Proof. See [15], Example 5 for characteristic zero and [13], Proposition 36 for prime characteristic. □

Definition 2.13. Let $\mathfrak{a}_\bullet$ be a graded sequence of monomial ideals. That is, suppose $\mathfrak{a}_r\mathfrak{a}_s \subseteq \mathfrak{a}_{r+s}$ for all $r, s \in \mathbb{Z}^+$. We define $\Gamma(\mathfrak{a}_\bullet)$ as the closure of the ascending union of the sets $\frac{1}{2^n}\Gamma(\mathfrak{a}_{2^n})$.

Following the proof of [5], Theorem 1.4 and the terminology of [16], we also define the *limiting polytope* of an ideal $I \subseteq R = K[x_1, \ldots, x_n]$.

Definition 2.14. Let $>$ be a monomial order on R. We set $\Gamma_>(I) = \Gamma(\mathfrak{a}_\bullet)$, where $\mathfrak{a}_n = \text{in}_>(I^n)$.

2.3. **Mixed Multiplicities and the Demailly-Pham Invariant.** To begin, we recall the definition of the mixed multiplicity symbol $e(I_1, \ldots, I_d; M)$.

Definition 2.15. Let M be a finite-length R-module. We let $\lambda_R(M)$ denote the length of M as an R-module.

Theorem 2.16 ([21], Theorem 17.4.2)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I_1, \ldots, I_k$ ideals of R primary to $\mathfrak{m}$, and M a finitely-generated R-module. Then there exists a polynomial $P(n_1, \ldots, n_k)$ with rational coefficients and total degree at most $\dim R$ such that for all $n_1, \ldots, n_k \gg 0$, we have*

$$P(n_1, \ldots, n_k) = \lambda_R\left(\frac{M}{I_1^{n_1} \ldots I_k^{n_k} M}\right).$$

Remark 2.17. Suppose instead that S is a Noetherian ring, not necessarily local, and $\mathfrak{n}$ is any maximal ideal of S. If $I_1, \ldots, I_k$ are $\mathfrak{n}$-primary ideals in S, then $I_1^{n_1} \cdots I_k^{n_k}$ is $\mathfrak{n}$-primary for all $n_1, \ldots, n_k > 0$. Consequently, we have

$$\lambda_S\left(\frac{S}{I_1^{n_1} \cdots I_k^{n_k} S}\right) = \lambda_{S_\mathfrak{n}}\left(\frac{S_\mathfrak{n}}{I_1^{n_1} \cdots I_k^{n_k} S_\mathfrak{n}}\right)$$

for all $n_1, \ldots, n_k$, so Theorem 2.16 holds for $I_1, \ldots, I_k$ without assuming that S is local.

Definition 2.18 (Mixed Multiplicity)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let $I_1, \ldots, I_k$ be $\mathfrak{m}$-primary ideals of R. Let $Q(n_1, \ldots, n_k)$ denote the degree-d part of $P(n_1, \ldots, n_k)$. The coefficients of Q define the mixed multiplicities $e(I_1^{\langle d_1 \rangle}, \ldots, I_k^{\langle d_k \rangle}; M)$:

$$Q(n_1, \ldots, n_k) = \sum_{d_1 + \cdots + d_k = d} \binom{d}{d_1, \ldots, d_k}^{-1} e(I_1^{\langle d_1 \rangle}, \ldots, I_k^{\langle d_k \rangle}; M) \tag{2}$$

The expression $e(I_1^{\langle d_1 \rangle}, \ldots, I_k^{\langle d_k \rangle}; M)$ is shorthand for the expression $e(I_1, \ldots, I_1, \ldots, I_k, \ldots, I_d; M)$, where I_j is repeated d_j times.

Remark 2.19. Other authors, such as [21], have used the notation $e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M)$ instead. To avoid confusion with the Frobenius powers of the ideals I_j, we use angle brackets in the exponent.

Following [11], we now define the *Segre numbers* of an ideal.

Definition 2.20. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I denote an $\mathfrak{m}$-primary ideal. We define the jth Segre number of I as

$$e_j(I) = e(I^{\langle j \rangle}, \mathfrak{m}^{\langle d-j \rangle}; R).$$

Suppose instead $R = K[x_1, \ldots, x_d]$ is a polynomial ring over a field. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let I be an $\mathfrak{m}$-primary ideal. By Remark 2.17, the function $\lambda_R(R/I^{n_1}\mathfrak{m}^{n_2}R)$ is a polynomial for $n_1, n_2 \gg 0$. We may therefore define $e_j(I)$ in terms of this polynomial, and this definition agrees with the quantity $e_j(IR_\mathfrak{m})$.

We'll record a few basic properties of the Segre numbers in a polynomial ring.

Proposition 2.21. *Let $R = K[x_1, \dots, x_d]$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R, and let I be an $\mathfrak{m}$-primary ideal.*

(i) We have $e_0(I) = 1, e_1(I) = ord_{\mathfrak{m}}(I)$, and $e_d(I) = e(I)$.
(ii) The sequence $e_0(I), \dots, e_d(I)$ is log convex.
(iii) If $h_1, \dots, h_d$ are general 1-forms, then for all $0 \le j \le d$ we have $e_j(I) = e\left(\frac{I+(h_1,\dots,h_{d-j})}{(h_1,\dots,h_{d-j})}\right)$, where $e(-)$ denotes the usual Hilbert multiplicity.

Proof.
(i): Follows from (iii).
(ii): See [21], Theorem 17.7.2.
(iii): See [21], Corollary 17.4.7.

□

We will now define the Demailly-Pham invariant, first defined in [7] and named in [4].

Definition 2.22. Let $R = K[x_1, \dots, x_d]$, $\mathfrak{m}$ the homogeneous maximal ideal of R, and I an $\mathfrak{m}$-primary ideal. Then we set

$$DP(I) := \frac{1}{e_1(I)} + \dots + \frac{e_{d-1}(I)}{e_d(I)}.$$

This invariant satisfies a property similar to Theorem 2.28.

Proposition 2.23. *Assume the setting of Definition 2.22, and let I_1, I_2 be $\mathfrak{m}$-primary ideals. Then $DP(I_1) \le DP(I_2)$ with equality if and only if $\overline{I_1} = \overline{I_2}$.*

Proof. This holds in much greater generality due to [4], Corollary 11. We need only that R is quasi-unmixed. □

In [4, 7], the authors define this invariant in the case $R = \mathcal{O}_d$, the ring of germs of analytic functions $f : (\mathbb{C}^d, 0) \to \mathbb{C}$. In this setting, the lct of an ideal $I = (g_1, \dots, g_r) \subseteq \mathcal{O}_d$ is defined as the supremum of all $s \in \mathbb{R}^+$ such that the function $(|g_1|^2 + \dots + |g_r|^2)^{-s}$ is locally integrable at 0. In this setting, we have the following result due to Bivià-Ausina.

Theorem 2.24 ([4], Theorem 13)**.** *Let $R = (\mathcal{O}_n, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal. Let I^0 denote the smallest integrally-closed monomial ideal containing I. Then the following are equivalent:*

(i) There exist integers $d_1, \dots, d_n$ such that $\overline{I} = (x_1^{d_1}, \dots, x_n^{d_n})$
(ii) $\operatorname{lct}(I^0) = DP(I)$
(iii) $\operatorname{lct}(I) = DP(I)$ *and* $\operatorname{lct}(I) = \operatorname{lct}(I^0)$.

We will demonstrate that the relationship between lct and DP among $\mathfrak{m}$-primary ideals is the same in $\mathcal{O}_d$ and $\mathbb{C}[x_1, \dots, x_d]$.

Proposition 2.25. *Let $R = \mathbb{C}[x_1, \dots, x_d]$ and let $\mathfrak{m} = (x_1, \dots, x_d)$, and let $\mathcal{O}_d$ denote the ring of germs of analytic functions $f : (\mathbb{C}^d, 0) \to \mathbb{C}$. Then $\mathcal{O}_d$ is local with maximal ideal $\mathfrak{m}\mathcal{O}_d$ and there is a bijective correspondence*

$$\{\mathfrak{m} - \textit{primary ideals in } R\} \to \{\mathfrak{m}\mathcal{O}_d - \textit{primary ideals in } \mathcal{O}_d\} \quad \textit{given by} \quad I \mapsto I\mathcal{O}_d. \tag{3}$$

Moreover, this correspondence preserves both DP and lct.

Proof. The fact that $\mathfrak{m}\mathcal{O}_d$ is the maximal ideal of $\mathcal{O}_d$ is well-known and follows from the identification of $\mathcal{O}_d$ with the ring of convergent power series $\mathbb{C}\{x_1, \dots, x_d\}$. This identification will also allow us to demonstrate that the correspondence 3 is bijective.

Since $R \to \mathbb{C}[[x_1, \dots, x_d]]$ is faithfully flat, for any $\mathfrak{m}$-primary ideal $I \subseteq R$ we have

$$I \subseteq I\mathcal{O}_d \cap R \subseteq I\mathbb{C}[[x_1, \dots, x_d]] \cap R = I,$$

so 3 is injective. On the other hand, given $J \subseteq \mathcal{O}_d$ primary to $\mathfrak{m}\mathcal{O}_d$, there exists an integer n such that $\mathfrak{m}^n\mathcal{O}_d \subseteq J$, so there exists a generating set for J consisting of polynomials of total degree at most n. It follows that J is extended from R, hence 3 is surjective.

To see that the $DP(I) = DP(I\mathcal{O}_d)$, it suffices to note that

$$\lambda_R \left(\frac{R}{I^{n_1}\mathfrak{m}^{n_2}R} \right) = \lambda_{\mathcal{O}_d} \left(\frac{\mathcal{O}_d}{I^{n_1}\mathfrak{m}^{n_2}\mathcal{O}_d} \right)$$

for all n_1, n_2. The fact that $\operatorname{lct}(I) = \operatorname{lct}(I\mathcal{O}_d)$ is [18], Theorem 1.2. □

2.4. **Integral Closure of Ideals.**

Definition 2.26. Let I be an ideal in a ring R. An element $r \in R$ is integral over I if there exists an integer n and elements $a_1, \dots, a_n, a_i \in I^i$ such that

$$r^n + a_1 r^{n-1} + \cdots + a_n.$$

We then define the integral closure $\overline{I}$ of I as the set of elements $r \in R$ which are integral over I.

Those hoping for an exhaustive discussion of the integral closure of ideals should consult [21]. For now, we will list some basic properties of $\overline{I}$.

Proposition 2.27 (Properties of the Integral Closure, [21] Chapter 1)**.** *Let R be a ring and $I \subseteq R$ an ideal. Let $\varphi : R \to S$. Then we have*

- *(i): $\overline{I}$ is an ideal.*
- *(ii): $\overline{(\overline{I})} = \overline{I}$.*
- *(iii): $\overline{I}S \subseteq \overline{IS}$.*
- *(iv): If $J \subseteq S$ is an ideal, then $\varphi^{-1}(\overline{J}) = \overline{\varphi^{-1}(J)}$.*
- *(v): For any multiplicatively-closed subset $W \subseteq R$, we have $W^{-1}\overline{I} = \overline{W^{-1}I}$.*
- *(vi): The integral closure of a monomial ideal $\mathfrak{a}$ in a polynomial ring $K[x_0, \dots, x_n]$ is generated by the set $x^\alpha : \alpha \in \Gamma(\mathfrak{a})$.*
- *(vii): If φ is faithfully flat or an integral extension, then $\overline{I}S \cap R = \overline{I}$.*

Integral closure is an operation which respects many numerical invariants we are interested in this paper.

Theorem 2.28 ([21], Proposition 11.2.1, Theorem 11.3.1)**.** *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $I \subseteq J$ two $\mathfrak{m}$-primary ideals. Then $e(I) = I(J)$ if and only if $\overline{I} = \overline{J}$.*

The same result, of course, holds in the case that $(R, \mathfrak{m})$ is instead standard-graded.

Proposition 2.29. *Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal. Then $c(I) = c(\overline{I})$.*

Proof. For characteristic zero, see [18], Property 1.15. For positive characteristic, see [22], Proposition 2.2 (6). □

2.5. **Essential Dimension.**

Definition 2.30 (Essential Dimension)**.** Let $J \subseteq R = K[x_1, \dots, x_d]$ be a homogeneous ideal. The essential dimension $\mathfrak{e}(J)$ is equal to the minimal r for which there exist linear forms $\ell_1, \dots, \ell_r$ such that J is extended from $I \subseteq K[\ell_1, \dots, \ell_r]$.

We have the following result.

Proposition 2.31 ([1], Proposition 3.3)**.** *Let k be an algebraically-closed field, $R = k[x_0, \dots, x_n]$, and $J \subseteq R$ a homogeneous ideal. Set $r = \operatorname{codim}(J)$. Let $L = (\ell_{r+1}, \dots, \ell_n)$, where the ℓ_i are chosen generally. For $r \leq t \leq n$, set $L_t = (\ell_{t+1}, \dots, \ell_n)$ and $J_t = \frac{J+L_t}{L_t}$. Then for all $r \leq t \leq n$, we have $\mathfrak{e}(J_t) = \max(t+1, \mathfrak{e}(J))$.*

3. The Limiting Polytope

3.1. **Complete Intersections in Positive Characteristic.** In this subsection, we prove [16, Theorem 1.1] over a field of characteristic $p > 0$. While the main argument is nearly identical, some intermediate lemmas must be weakened. In particular, [16, Lemma 3.6] is false in positive characteristc, which is evident by considering $\operatorname{gin}(x^p, y^p) \subseteq K[x, y]$ for an infinite field K of characteristic p. For the sake of self-containedness, we will sketch the entire adapted argument here.

Lemma 3.1. *Let K be a field, $R = K[x_1, \dots, x_n]$, and $J \subseteq R$ a homogeneous ideal. Let $1 \leq j \leq n$ and define $\pi_j : R \to R/(x_{j+1}, \dots, x_n) \cong K[x_1, \dots, x_j]$. If $>$ denotes the reverse lexicographic order, then*

$$\operatorname{in}_> \pi_j(J) = \pi_j(\operatorname{in}_>(J)).$$

Proof. Let $f \in J$ be a homogeneous element. Write $f = g + h$, where $h \in (x_{j+1}, \dots, x_n)$ and $\frac{\partial g}{\partial x_i} = 0$ for all $j+1 \leq i \leq n$. If $g = 0$, then $\pi_j(f) = 0$. If $g \neq 0$, then $\operatorname{in}_>(f) = \operatorname{in}_>(g)$. In both cases, we have $\pi_j(\operatorname{in}_>(f)) = \operatorname{in}_>(\pi_j(f))$. □

Definition 3.2. Let K be an infinite field. Let $R = K[x_1, \dots, x_n]$ and let $>$ denote the reverse lexicographic order. Let $I = (f_1, \dots, f_n)$ be a complete intersection ideal, where f_i is homogeneous of degree d_i and $d_1 \leq \cdots \leq d_n$. For $1 \leq j \leq n$, let $I_j := (f_1, \dots, f_j)$. For $1 \leq j \leq n$, let $\pi_j : R \to R/(x_{j+1}, \dots, x_n) \cong K[x_1, \dots, x_j]$ denote the projection map. Let $\varphi_m \in \mathrm{GL}_n(K)$ be a general linear transformation such that $(\varphi_m^{-1})^*(x_{j+1}, \dots, x_n)$ is regular on R/I_j^m and $\operatorname{in}_>(\varphi_m^* I_j^m) = \operatorname{gin}_>(I_j^m)$ for all $1 \leq j \leq n$.

Lemma 3.3. *Assume the setting of Definition 3.2. For all $1 \leq j \leq n, m > 0$, we have* $\operatorname{gin}_>(I^m) = (\operatorname{in}_>(\pi_j(\varphi_m^* I_j^m)))R$.

Proof. Since I_j is a complete intersection, I_j^m is Cohen-Macaulay for all $m > 0$, hence $\operatorname{codim}(I_j^m) = \operatorname{depth}(I_j^m) = j$. Consequently, by [14, Lemma 3.1], the generators of $\operatorname{gin}_>(I_j^m)$ are contained in $K[x_1, \dots, x_j]$, so $\pi_j(\operatorname{gin}_>(I_j^m))R = \operatorname{gin}_>(I_j^m)$. By Lemma 3.1, we have

$$\operatorname{gin}_>(I_j^m) = \pi_j(\operatorname{gin}_>(I_j^m))R = \pi_j(\operatorname{in}_>(\varphi_m^* I_j^m))R = (\operatorname{in}_>(\pi_j(\varphi_m^* I_j^m)))R.$$

□

Lemma 3.4. *Assume the setting of Definition 3.2. Then for all $1 \leq j \leq n, m > 0$ we have $x_j^{(m+j-1)} \in \operatorname{gin}_>(I_j^m)$.*

Proof. By Lemma 3.3, it suffices to prove the result when $j = n$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. By Lemma 3.9, we have $\mathfrak{m}^{d_n} \subseteq \overline{I}$. By the Briançon-Skoda theorem, we have $\mathfrak{m}^{(m+n-1)d_n} \subseteq \overline{I}^{m+n-1} \subseteq I^m$. It follows that

$$x_n^{(m+n-1)d_n} \in \mathfrak{m}^{(m+n-1)d_n} = \varphi_m^* \mathfrak{m}^{(m+n-1)d_n} \subseteq \operatorname{in}_>(\varphi_m^* I^m) = \operatorname{gin}_>(I^m).$$

□

Proposition 3.5. *Assume the setting of Definition 3.2. Let $\mathfrak{a}_\bullet$ be the graded system of ideals given by $\mathfrak{a}_m = \operatorname{gin}(I^m)$. Then*

$$\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, (d_1, 0, \dots, 0), (0, d_2, 0, \dots, 0), \dots, (0, \dots, 0, d_n)\right). \tag{4}$$

Proof. By Lemma 3.4, we have

$$\operatorname{conv}\left(\vec{0}, (d_1, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_n)\right) \subseteq \overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}. \tag{5}$$

As $e(I) = d_1 \ldots d_n$, we also have $\operatorname{vol}(\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}) = (d_1 \ldots d_n)/n!$ by [VOL=MULT]. It follows that the containment in Equation (5) is in fact equality. ☐

Corollary 3.6. *Assume the setup of Definition 3.2 and let $r < n$. Let $J = (f_1, \ldots, f_r)$ and for $m > 0$ set $\mathfrak{a}_m := \operatorname{gin}_>(J^m)$. Then we have*

$$\operatorname{conv}\left(\vec{0}, (d_1, 0, \ldots, 0), (0, d_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_r, 0, \ldots, 0)\right) = \overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)}. \tag{6}$$

Proof. Follows from and Lemma 3.3 and Proposition 3.5. ☐

3.2. **F-Pure Thresholds and the Demailly-Pham Invariant.** In this subsection, we require an asymptotic version of Theorem 2.24 in arbitrary characteristic. Only minor refinements of Bivià-Ausina's arguments are needed.

Lemma 3.7. *Let K be an infinite field, $R = K[x_1, \ldots, x_n]$, and I an $\mathfrak{m}$-primary homogeneous ideal. If $>$ denotes the reverse lexicographic order, then for all $1 \leq j \leq n$ we have*

$$\lim_{t \to \infty} \frac{e_j(\operatorname{gin}_>(I^t))}{t^j} = e_j(I).$$

This result was shown by Bivià-Ausina [4, Theorem 4] in the related setting where $R = \mathcal{O}_n$ and $>$ denotes the negative lexicographic order.

Proof. Without loss of generality, we first extend K to an uncountably infinite field; this changes neither the hypothesis nor the conclusion.

If $J \subseteq R$ is an $\mathfrak{m}$-primary homogeneous ideal and $h_1, \ldots, h_n$ is a sequence of linear forms, we say that a $h_1, \ldots, h_n$ *computes* $e_\bullet(J)$ if for $1 \leq j \leq n$, we have

$$e_j(J) = e\left(\frac{J + (h_1, \ldots, h_{n-j})}{(h_1, \ldots, h_{n-j})}\right).$$

For $m > 0$, we define:

- U_m is the open subset of $\operatorname{GL}_n(K)$ such that $\operatorname{in}_>(\varphi^* I^m)$ is constant for all $\varphi \in U_m$, such that U_m meets nontrivially the unipotent subgroup of upper triangular matrices with ones along the diagonal, and such that U is fixed by the Borel subgroup of upper-triangular matrices.
- V_m is the open subset of $\operatorname{GL}_n(K)$ for which $\varphi^* x_n, \ldots, \varphi^* x_1$ computes $e_\bullet(I^m)$.
- W_m is the open subset of $\operatorname{GL}_n(K)$ for which $\varphi^* x_n, \ldots, \varphi^* x_1$ computes $e_\bullet(\psi^*(I^m))$, where $\psi \in U_m$ is arbitrary.

Nonemptiness of U_m is [8, Theorem 15.18]. Nonemptiness of V_m, W_m follows from Proposition 2.21. Since K is uncountable, we may choose $\varphi \in \bigcap_{m>0}(U_m \cap V_m \cap W_m)$. Set $J = (\varphi^{-1})^* I$, and for $1 \leq j \leq n$, let $\pi_j : R \to R/(x_{j+1}, \ldots, x_n)$. We then have

$$\begin{aligned} e_j(I) = e(\pi_j(J)) &= \lim_{m \to \infty} \frac{1}{m^j} e(\pi_j(J^m)) \\ &= \lim_{m \to \infty} \frac{1}{m^j} e(\operatorname{in}_>(\pi_j(I^m))) = \lim_{m \to \infty} \frac{1}{m^j} e(\pi_j(\operatorname{in}_>(I^m))) \\ &= \lim_{m \to \infty} \frac{1}{m^j} e_j(\operatorname{in}_>(J^m)). \end{aligned}$$

The first and fifth equalities follow the the fact that $x_1, \ldots, x_n$ computes $e_\bullet(J^m)$ and $e_\bullet(\operatorname{in}_>(J^m))$ for all $m \geq 1$. The second follows from the equality $e(\pi_j(J^m) = e(\pi_j(J)^m) = m^j e(\pi_j(J))$. The third is from [19, Corollary 1.13], and the fourth is from Lemma 3.1. ☐

Definition 3.8. Let K be an infinite field, $R = K[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and let $\mathfrak{a}_\bullet$ be a graded system of $\mathfrak{m}$-primary ideals. We define:

- The asymptotic Segre numbers: $e_j(\mathfrak{a}_\bullet) = \liminf_m \frac{e_j(\mathfrak{a}_m)}{m^j}$.
- The asymptotic Demailly-Pham invariant: $DP(\mathfrak{a}_\bullet) = \frac{1}{e_1(\mathfrak{a}_\bullet)} + \cdots + \frac{e_{n-1}(\mathfrak{a}_\bullet)}{e_n(\mathfrak{a}_\bullet)}$
- The asymptotic singularity threshold: $c(\mathfrak{a}_\bullet) = \liminf_m mc(\mathfrak{a}_m)$.

Before we prove our asymptotic version of Theorem 2.24, we require the following standard facts.

Lemma 3.9. *Let L be a field, $S = L[x_1, \dots, x_n]$, and $J \subseteq S$ an $\mathfrak{m}$-primary homogeneous ideal generated by forms of degree $\leq d$. Then $\mathfrak{m}^d \subseteq \overline{J}$.*

Proof. We first prove the result in the case that L is infinite. First, choose forms $f_1, \dots, f_n$ from among the generators of J such that $(f_1, \dots, f_n)$ is $\mathfrak{m}$-primary. If $h_1, \dots, h_n$ are general linear forms, then

$$J' := (h_1^{d-\deg(f_1)} f_1, \dots, h_n^{d-\deg(f_n)} f_n)$$

is an $\mathfrak{m}$-primary $(d_n, \dots, d_n)$-complete intersection contained in J. As $J' \subseteq \mathfrak{m}^d$ and $e(J)' = d^n = e(\mathfrak{m}^d)$, we have $\mathfrak{m}^d = \overline{\mathfrak{m}^d} \subseteq \overline{J'} \subseteq \overline{J}$ by Theorem 2.28.

Now, let L be an arbitrary field, and set $S' = \overline{L}[x_1, \dots, x_n]$. By Proposition 2.27 (vii) and the infinite field case, we have $\overline{J} = \overline{JS'} \cap S \supseteq \mathfrak{m}^d S' \cap S = \mathfrak{m}^d$. $\square$

Lemma 3.10. *Let L be a field and $S = L[x_1, \dots, x_n]$. Let $J = (f_1, \dots, f_n)$ be a complete intersection where $\deg f_i = d_i$ and $d_1 \leq \cdots \leq d_n$. Then we have the following:*

(i) If L is infinite, then for a general hyperplane section $H \subseteq \operatorname{Spec} R$, we have $e(I|_H) = d_1 \cdots d_{n-1}$.

(ii) With no assumption on $|L|$, we have $DP(I) = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$.

Proof. For (i), we note that for a general hyperplane section H, we have that $(f_1, \dots, f_{n-1})|_H$ is $\mathfrak{m}$-primary. By Lemma 3.9, we have $(\mathfrak{m}|_H)^{d_{n-1}} \subseteq \overline{(f_1, \dots, f_{n-1})|_H}$. As $f_n \in (\mathfrak{m}|_H)^{d_{n-1}}$, we have $\overline{(f_1, \dots, f_{n-1})|_H} = \overline{J|_H}$. Consequently, $e(J|_H) = e(\overline{J|_H}) = d_1 \cdots d_{n-1}$.

For (ii), we note that $DP(J)$ is invariant under extension of the base field, so it suffices to consider the case of an infinite field. But then the result follows from (i) and Proposition 2.21 (iii). $\square$

Corollary 3.11. *Let K be an infinite field, $R = K[x_1, \dots, x_n]$, and I an $\mathfrak{m}$-primary homogeneous ideal. Then $DP(I) \leq c(I)$. Moreover, let $>$ denote the reverse lexicographic order. Suppose $DP(I) = c(I)$. Letting $\mathfrak{a}_m := \operatorname{gin}_>(I^m)$, we have*

$$\overline{\mathbb{R}^n_{\geq 0} \setminus \Gamma(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, (e_1(I), 0, \dots, 0), \left(0, \frac{e_2(I)}{e_1(I)}, 0, \dots, 0\right), \dots, \left(0, \dots, 0, \frac{e_n(I)}{e_{n-1(I)}}\right)\right). \tag{7}$$

Proof. By Lemma 3.7 we have $DP(\mathfrak{a}_\bullet) = DP(I)$ and by Proposition 2.7 we have $c(\mathfrak{a}_\bullet) \leq c(I)$. Let χ denote the characteristic function of $\Gamma := \Gamma_>(\varphi^* I)$. Since Γ is convex, $-\chi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Let $\mu = \inf_t : (t, \dots, t) \in \Gamma$. Since $-\chi$ is proper, there exists a subgradient v to $-\chi$ at $\vec{\mu} := (\mu, \dots, \mu)$. Let H^- denote the set $\{x \in \mathbb{R}^n : \langle v, (x - \vec{\mu}) \rangle \leq 0\}$. As

$$\langle v, (x - \vec{\mu}) \rangle \leq -\chi(x) + \chi(\vec{\mu}) = 1 - \chi(x)$$

for all $x \in \mathbb{R}_n$, we have $\Gamma \subseteq H^-$. Since Γ is closed under translation by elements of $R^n_{\geq 0}$ and the complement of Γ in $\mathbb{R}^n_{\geq 0}$ is bounded, the same is true for H^-. Consequently, the complement of H^- in $\mathbb{R}^n_{\geq 0}$ is a simplex $\operatorname{conv}(0, (b_1, 0, \dots, 0), \dots, (0, \dots, 0, b_n))$.

Define a graded system of monomial ideals $\mathfrak{b}_\bullet$ by $\mathfrak{b}_m = x^\alpha : \alpha \in mH^-$. By Proposition 2.12, we have $c(\mathfrak{b}_\bullet) = c(\mathfrak{a}_\bullet)$. Since $\mathfrak{a}_m \subseteq \mathfrak{b}_m$ for all m, we have $DP(\mathfrak{a}_\bullet) \leq DP(\mathfrak{b}_\bullet)$ by Proposition 2.23, which implies $DP(I) \leq c(I)$.

Now suppose $DP(I) = c(I)$. Then we also have $DP(\mathfrak{a}_\bullet) = DP(\mathfrak{b}_\bullet)$. By [4, Proposition 10], we further have that $e_j(I) = e_j(\mathfrak{a}_\bullet) = e_j(\mathfrak{b}_\bullet)$ for all $1 \le j \le n$. In particular, $e_n(\mathfrak{a}_\bullet) = e_n(\mathfrak{b}_\bullet)$, so by [19, Theorem 2.12 and Lemma 2.13], we have $\mathrm{vol}(\mathbb{R}^n_{\ge 0} \setminus \Gamma(\mathfrak{a}_\bullet)) = (\mathbb{R}^n_{\ge 0} \setminus \Gamma(\mathfrak{b}_\bullet))$. Since $\Gamma(\mathfrak{a}_\bullet), \Gamma(\mathfrak{b}_\bullet)$ are closed and convex with positive volume, it follows that $\Gamma(\mathfrak{a}_\bullet) = \Gamma(\mathfrak{b}_\bullet)$.

Since the generic initial ideal is Borel-fixed, we have $b_1 \le \cdots \le b_n$. Consequently, we can compute $e_j(\mathfrak{b}_\bullet)$ in terms of the numbers b_j: we have

$$\overline{\left(x_1^{\lfloor mb_1 \rfloor}, \ldots, x_n^{\lfloor mb_n \rfloor}\right)} \subseteq \mathfrak{b}_m \subseteq \overline{\left(x_1^{\lceil mb_1 \rceil}, \ldots, x_n^{\lceil mb_n \rceil}\right)}.$$

It follows that $e_j(\mathfrak{b}_\bullet) = b_1 \cdots b_j$. As $e_j(I) = e_j(\mathfrak{b}_\bullet)$, the result follows. ☐

Remark 3.12. The condition Equation (7) is necessary to have $c(I) = DP(I)$, but not sufficient. By [16]

3.3. **Behavior of the Threshold Under Modifications.** In this section, fix the following notation.

Definition 3.13. Let K be a characteristic zero field, $R = K[x_1, \ldots, x_n]$, and let $\mathfrak{m}$ denote the homogeneous maximal ideal. Let $I \subseteq R$ be an $\mathfrak{m}$-primary homogeneous ideal. Write $I = I_1 + \cdots + I_r$, where I_j is generated by forms of degree d_j and $d_1 < \cdots < d_j$.

Let $A \subseteq K$ be a finitely-generated $\mathbb{Z}$-algebra and $J \subseteq A[x_1, \ldots, x_n]$ an ideal such that $JR = I$. Such a subring A can always be constructed by adjoining to $\mathbb{Z}$ the field coefficients appearing in a generating set for I. If μ is a maximal ideal of A, we let I_μ denote the image of J in $(A/\mu)[x_1, \ldots, x_n]$, and we write $I_\mu = I_{1,\mu} + \cdots + I_{r,\mu}$.

Lemma 3.14 ([2], Lemma 3.2). *Let $R = K[x_1, \ldots, x_n]$ and let $\mathfrak{m}$ denote the homogeneous maximal ideal. For any $e, t \in \mathbb{Z}^+$, we have*

$$(\mathfrak{m}^{[p^e]} : \mathfrak{m}^t) = \begin{cases} R & t \ge np^e - n + 1 \\ \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e - n + 1 - t} & t < np^e - n + 1 \end{cases}$$

More generally, we have the following.

Lemma 3.15. *Let $R = K[x_1, \ldots, x_n]$. Let v be a monomial valuation on R with $v(x_i) \ge 0$ for all $1 \le i \le n$. For $f \in R$, we define $v(f)$ to be the minimum of v over the monomials in the support of f. For $\lambda \in \mathbb{R}^+$, let $\mathfrak{a}_\lambda$ denote the ideal $\{f \in R : v(f) \ge \lambda\}$. Let $q \in \mathbb{Z}^+, \lambda \in \mathbb{R}^+$. Then we have*

$$((x_1^q, \ldots, x_n^q) : \mathfrak{a}_\lambda) = (x_1^q, \ldots, x_n^q) + \mathfrak{a}_{(q-1)v(x_1 \cdots x_n) - \lambda}. \tag{8}$$

Proof. The argument is the same as Lemma 3.14. Let $m \notin (x_1^q, \ldots, x_n^q)$ be a monomial. Then $m \mid (x_1 \cdots x_n)^{q-1}$, so $\mathfrak{a}_\lambda m \subseteq (x_1^q, \ldots, x_n^q)$ if and only if $v((x_1 \cdots x_n)^{q-1}) - v(m) \le \lambda$. We've shown that the two sides of Equation (8) contain the same monomials; both sides are monomial ideals, the result follows. ☐

Lemma 3.16 (Restriction to a Hyperplane I)**.** *Let K be a field of characteristic $p > 0$, let $R = K[x_1, \ldots, x_n]$, and $I \subseteq R$ a homogeneous ideal. For a hyperplane H cut out by a linear form ℓ, we let $I|_H$ denote the image of I in R/ℓ. In this case, we have*

$$\nu_{I|_H}(p^e) \le \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n-1)(p^e-1)+1}\}, \tag{9}$$

Conversely, if $|K| \ge p^e$, then there exists a hyperplane H such that

$$\nu_{I|_H}(p^e) \ge \max\{r : I^r \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{(n-1)(p^e-1)-(n-2)(p^{e-1})+1}\} \tag{10}$$

Corollary 3.17. *Assume the setup of Definition 3.13 and let $H \subseteq \operatorname{Spec} R$ be a hyperplane. For all μ such that I_μ is $\mathfrak{m}$-primary, we have $c(I_\mu) - c(I_\mu|_{H_\mu}) \ge \frac{1}{d_r}$, and consequently, $c(I) - c(I|_H) \ge 1/d_r$.*

Proof. Combining Lemma 3.14 and Lemma 3.16, we have

$$\nu_{I|_H}(p^e) \le \max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\}.$$

By Lemma 3.9, we have $\mathfrak{m}^{d_r} \subseteq \overline{I}$, so $\max\{s : \mathfrak{m}^{p^e} I^s \not\subseteq \mathfrak{m}^{[p^e]}\} \le \nu_I(p^e) - \left\lfloor \frac{p^e}{d_r} \right\rfloor$, so we have $\nu_{I|_H}(p^e) \le \nu_I(p^e) - \left\lfloor \frac{p^e}{d} \right\rfloor$. Dividing by p^e and taking the limit as $e \to \infty$ gives the result. □

Lemma 3.18. *Assume the setup of Definition 3.13. Suppose $r = 2$. Then we have*

$$c(I_\mu) = \frac{n}{d_2} + c(I_{1,\mu})\frac{d_2 - d_1}{d_2} \text{ for all } \mu \text{ and hence } c(I) = \frac{n}{d_2} + c(I_1)\frac{d_2 - d_1}{d_2}.$$

In particular, for any $\mu \in \max \operatorname{Spec} A, 1 \le s \le n$, we have $c(I_\mu) = \frac{s}{d_1} + \frac{n-s}{d_2}$ if and only if $c(I_{1,\mu}) = \frac{s}{d_1}$.

Proof. By Lemma 3.9, we have $\mathfrak{m}^{d_2} \subseteq \overline{I_\mu}$, so $I_\mu \subseteq I_{1,\mu} + \mathfrak{m}^{d_2} \subseteq \overline{I_\mu}$, so $c(I_\mu) = c(I_{1,\mu} + \mathfrak{m}^{d_2})$. Consequently, we have

$$\nu_{\overline{I}}(p^e) = \max\left\{a + b : I_{1,\mu}^a \mathfrak{m}_\mu^{bd_n} \not\subseteq \mathfrak{m}^{[p^e]}\right\} = \max\{a + b : J_p^a \not\subseteq (\mathfrak{m}_p^{[p^e]} : \mathfrak{m}_p^{bd_n})\}.$$

By Lemma 3.14, this is equivalent to

$$\nu_{\overline{I}}(p^e) = \max\{a + b : I_{1,\mu}^a \not\subseteq \mathfrak{m}^{[p^e]} + \mathfrak{m}^{np^e - n + 1 - bd_2}\} = \max_{0 \le a \le \nu_{I_{1,\mu}}(p^e)} a + \frac{np^e - n + 1 - ad_1}{d_2}. \tag{11}$$

The quantity being maximized in Equation (11) is an increasing function of a, so

$$\nu_{\overline{I}}(p^e) = \frac{np^e - n + 1}{d_2} + \nu_{I_{1,\mu}}(p^e)\frac{d_2 - d_1}{d_2}.$$

Dividing by p^e and letting $e \to \infty$, we obtain

$$c(I_\mu) = \frac{n}{d_2} + c(I_{1,\mu})\frac{d_2 - d_1}{d_2}.$$

For characteristic zero, we compute

$$c(I) = \sup_\mu c(I_\mu) = \sup_\mu \left(\frac{n}{d_2} + c(I_{1,\mu})\frac{d_2 - d_1}{d_2}\right) = \frac{n}{d_2} + c(I)\frac{d_2 - d_1}{d_2}.$$

□

4. Proof of Theorem 4.14

In this section, we will prove Theorem 4.14. Our proof techniques work in arbitrary characteristic, and we will begin in the case of a complete intersection.

4.1. The Case of an $\mathfrak{m}$-Primary Complete Intersection.

Definition 4.1. We assume a setup similar to Definition 3.13. Let K be an algebraically-closed field. Let $a_1, \ldots, a_r, d_1, \ldots, d_r \in \mathbb{Z}^+$. For $1 \le i \le r$, let $\mathbf{x}_i$ denote the tuple of variables $x_{i,1}, \ldots, x_{i,a_i}$, and let $R = K[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. Let $I \subseteq R$ be a complete intersection of the form $(f_{1,1}, \ldots, f_{r,a_r})$ such that $f_{i,j}$ is a d_i-form. For $1 \le j \le r$, write $I_j = (f_{j,1}, \ldots, f_{j,a_j})$. Let v denote the monomial valuation with $v(x_{i,j}) = 1/d_i$. Finally, we let $\mathfrak{D}$ denote the ideal $\overline{(\mathbf{x}_1^{d_1}, \ldots, \mathbf{x}_r^{d_r})}$, which coincides with the set of elements of valuation $v(-) \ge 1$.

Definition 4.2. Assume the setup of Definition 4.1. We the following condition on the ideal I:

$$(\dagger) \qquad \begin{gathered} I_1 \text{ is extended from } K[\mathbf{x}_1] \text{ and} \\ \text{For } 2 \le i \le r, \frac{I_i + (\mathbf{x}_1)}{(\mathbf{x}_1)} \subseteq \frac{\mathfrak{D} + (\mathbf{x}_1)}{(\mathbf{x}_1)}. \end{gathered}$$

Lemma 4.3. *Assume the setup of Definition 4.1 and suppose $c(I) = DP(I)$. Let $\ell \in R$ be a general linear form and let H denote the zero locus of ℓ. Then $c(H, I|_H) = DP(I|_H)$.*

Proof. By Lemma 3.10, Corollary 3.11, and Corollary 3.17, we have

$$\frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r} = DP(I|_H) \leq c(I|_H) \leq \frac{a_1}{d_1} + \cdots + \frac{a_r - 1}{d_r}.$$

□

Our first step in the proof of Theorem 4.14 is to show that for any complete intersection with $DP(I) = c(I)$, there exists $\varphi \in \mathrm{GL}_n(K)$ such that $\varphi^*(I)$ satisfies Equation (†). Before we begin, we take a moment to briefly clarify the unusual structure of our induction. Lemma 4.4 for an integer r depends on Lemma 4.4 and Theorem 4.14 for $r-1$, whereas Theorem 4.14 for an integer r depends on Lemma 4.4 for r and Theorem 4.14 for $r-1$.

Lemma 4.4. *Assume the setup of Definition 4.1. Suppose $c(I) = DP(I)$. Then there exists $\varphi \in \mathrm{GL}_n(K)$, depending only on $I_1, \ldots, I_{r-1}$, such that $\varphi^* I$ satisfies Equation (†).*

Proof. Note that $DP(I) = a_1/d_1 + \cdots + a_r/d_r$. We induct on r, and we also assume that Theorem 4.14 holds for $r' < r$. The base case $r = 1$ is trivial: $\overline{I} = \mathfrak{m}^{d_1}$ by Theorem 2.28, so we may take $\rho = \mathrm{id}$.

Now suppose $r > 1$. Let L be an ideal of R generated by $a_3 + \cdots + a_d$ general linear forms. Since I is a complete intersection, $I_1 + I_2 + L$ is $\frac{\mathfrak{m}}{L}$-primary, so by Lemma 3.9, we have

$$\overline{\frac{I_1 + I_2 + L}{L}} \supseteq \frac{I_1 + \mathfrak{m}^{d_2} + L}{L} \supseteq \frac{I + L}{L},$$

so we have $c(R/L, \frac{I+L}{L}) = c(R/L, \frac{I_1+I_2+L}{L})$. Consequently, by repeated application of Corollary 3.17, we have

$$c(I) \geq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) + \frac{a_3}{d_3} + \cdots + \frac{a_r}{d_r}. \tag{12}$$

Assuming $c(I) = DP(I)$, we have

$$\frac{a_1}{d_1} + \frac{a_2}{d_2} \leq c\left(R/L, \frac{I_1 + I_2 + L}{L}\right) \leq \frac{a_1}{d_1} + \frac{a_2}{d_2},$$

where the left-hand side is by [DP-lowerbound] and the right-hand side is by Equation (12). Both inequalities are therefore equalities, so by Lemma 3.18 we have

$$c\left(R/L, \frac{I_1 + L}{L}\right) = \frac{a_1}{d_1}.$$

By Theorem [1], Theorem 3.17, we have $\mathfrak{e}\left(\frac{I_1+L}{L}\right) = a_1$, hence by Proposition 2.31 we have that $\mathfrak{e}(I_1) = a_1$. Consequently, there exists $\varphi \in GL_n(K)$ depending only on I_1 such that $\varphi^* I_1$ is extended from $K[\mathbf{x}_1]$. Now, let $\succ$ denote the monomial partial order induced by the monomial valuation $w(x_{1,i}) = 0$ and $w(x_{i,j}) = 1$ for $i \geq 2$. For $2 \leq i \leq r, 1 \leq j \leq a_i$, let $g_{i,j} := \mathrm{in}_\succ(\varphi^* f_{i,j})$. Since $\varphi^* I$ is a complete intersection, we have $\varphi^* f_{i,j} \notin \sqrt{\varphi^* I_1} = (\mathbf{x}_1)$, hence $g_{i,j} \notin (\mathbf{x}_1)$ and moreover $\varphi^* f_{i,j} - g_{i,j} \in (\mathbf{x}_1)$. Observe that

$$\mathrm{in}_\succ(\varphi^* I) \supseteq \varphi^* I_1 + \varphi^* \mathrm{in}_\succ(I_2 + \cdots + I_r) \supseteq \varphi^* I_1 + (g_{2,1}, \ldots, g_{r,a_r}). \tag{13}$$

Let I' denote the right-hand side of Equation (13). Because $g_{i,j}$ and $\varphi^* f_{i,j}$ have the same image modulo $(\mathbf{x}) = \sqrt{\varphi^* I_1}$, the ideal I' is a complete intersection. In particular, I' is a complete intersection of type $(\underbrace{d_1, \ldots, d_1}_{a_1}, \ldots, \underbrace{d_r, \ldots, d_r}_{a_r})$.

By Lemma 3.10 and Proposition 2.7, we have

$$DP(I) = DP(I') \leq c(I') \leq c(\mathrm{in}_\succ(\varphi^* I)) \leq c(I) = DP(I). \tag{14}$$

As $\overline{\varphi^* I_1} = (\mathbf{x}_1)^{d_1}$, we have $c(\varphi^*(I_1)) = a_1/d_1$. Since $\varphi^* I_1$ and $(g_{2,1}, \ldots, g_{r,a_r})$ are defined in terms of disjoint sets of variables, we have by [23], Theorem 2.4 (1) that

(15)
$$c(R, I') = c(K[\mathbf{x}_1], \varphi^* I_1) + c(K[\mathbf{x}_2, \ldots, \mathbf{x}_r], (g_{2,1}, \ldots, g_{r,a_r})) = \frac{a_1}{d_1} + c(K[\mathbf{x}_2, \ldots, \mathbf{x}_r], (g_{2,1}, \ldots, g_{r,a_r})).$$

It follows from Equations (14) and (15) that $(g_{2,1}, \ldots, g_{r,a_r})$, which is a complete intersection in $K[\mathbf{x}_2, \ldots, \mathbf{x}_r]$ also has $DP = \mathrm{lct}$. By induction, there exists $\psi \in \mathrm{GL}_{n-a_1}(K)$ depending only on $(g_{2,1}, \ldots, g_{r-1,a_{r-1}})$ such that $\psi^*(g_{2,1}, \ldots, g_{r-1,a_{r-1}}) \subseteq \overline{(\mathbf{x}_2^{d_2}, \ldots, \mathbf{x}_r^{d_r})}$. Lastly, we define $\rho := \begin{bmatrix} \mathrm{id}_{a_1} & 0 \\ 0 & \psi \end{bmatrix} \circ \varphi$, a transformation which depends only on $I_1, \ldots, I_{r-1}$, and we claim that $\rho^* I$ satisfies Equation (†). By construction, $\rho^* I + \psi^*(g_{1,1}, \ldots, g_{r,a_r})R$ satisfies Equation (†). Since $g_{i,j} - \varphi^* f_{i,j} \in (\mathbf{x}_1)$ for all $2 \leq i \leq r, 1 \leq j \leq a_i$, we have $\rho^*(I_2 + \cdots + I_r) \subseteq \psi^*(g_{1,1}, \ldots, g_{r,a_r})R + (\mathbf{x}_1)$, which proves that $\rho^* I$ also satisfies Equation (†). ☐

The following two lemmas will combine with Lemma 4.4 to prove Theorem 4.14 in the case of a complete intersection.

Lemma 4.5. *Assume the setting of Definition 4.1. Suppose I satisfies Equation (†) and $I \not\subseteq \mathfrak{D}$. Then there exists an ideal J and an integer $2 \leq m \leq r-1$ such that:*

1. *(1) $J = (\mathbf{x}_1)^{d_1} + (h_{2,1}, \ldots, h_{r-1,a_{r-1}}) + \cdots + J_{r-1} + \mathfrak{m}^{d_r}$, where $h_{i,j}$ is homogeneous of degree d_j.*
2. *(2) For all $1 \leq j \leq a_m$, there exists $h'_{m,j}$ such that $h'_{m,j} \in K[\mathbf{x}_m], h'_{m,j} - h_{m,j} \in (\mathbf{x}_1)$, and no monomial summand of $h_{m,j} - h'_{m,j}$ is contained in $\mathfrak{D}$. Moreover, there exists some j such that $h_{m,j} - h'_{m,j} \neq 0$.*
3. *(3) For all $2 \leq i \leq r-1, i \neq m, 1 \leq j \leq a_i$, we have $h_{i,j} \in K[\mathbf{x}_i]$.*
4. *(4) $(h_{2,1}, \ldots, h_{r-1,a_{r-1}})$ is a complete intersection mod $(\mathbf{x}_1)$.*
5. *(5) $c(J) \leq c(I)$*

Proof. Suppose I satisfies Equation (†), but $I \not\subseteq \mathfrak{D}$. We will construct a simpler ideal J such that $J \not\subseteq \mathfrak{D}, DP(I) = DP(J)$, and $c(J) \leq c(I)$.

We define a set T_0 which measures the failure of I to be contained in $\mathfrak{D}$. First, for $b = (e_{1,1}, \ldots, e_{r,a_r}) \in \mathbb{R}^n$ we define $\pi(b) = \left(\sum_{j=1}^{a_1} e_{1,j}, \ldots, \sum_{j=1}^{a_r} e_{r,j}\right) = (\pi_1(b), \ldots, \pi_r(b))$. We then define the sets

(16)
$$S_0^{i,j} := \left\{\pi(b) : x^b \in \mathrm{supp}(f_{i,j}), \pi_1(b) \neq 0\right\}, \quad S_0^i = \bigcup_{j=1}^{a_i} S_0^{i,j}, \quad S_0 = \bigcup_{i=2}^{r-1} S_0^i.$$

Note that the condition $x^b \notin \mathfrak{D}$ is equivalent to the condition $v(x^b) < 1$, so we define

(17)
$$T_0^{i,j} = \{(u_1, \ldots, u_r) \in S_0^{i,j} : \sum_{i=1}^{r} \frac{u_i}{d_i} < 1\}, \quad T_0^i = \bigcup_{j=1}^{a_i} T_0^{i,j}, \quad T_0 = \bigcup_{i=2}^{r-1} T_0^i.$$

Since I satisfies Equation (†), we also have $u_1 > 0$ for all $(u_1, \ldots, u_r) \in S_0$. Let

$$t_0 := \max_{(u_1, \ldots, u_r) \in S_0} \frac{1 - u_1/d_1 - \cdots - u_r/d_r}{u_1}.$$

Note that the quantity $\frac{1-u_1/d_1-\cdots-u_r/d_r}{u_1}$ is positive precisely when $(u_1, \ldots, u_r) \notin \mathfrak{D}$, so $t_0 > 0$ and the elements of S_0 achieving this maximum are all in T_0. Define $w_0 : \mathbb{Z}^r \to \mathbb{Q}$ by

$$w_0(u_1, \ldots, u_r) = \left(-\frac{1}{d_1} - t_0\right) u_1 - \frac{u_2}{d_2} - \cdots - \frac{u_r}{d_r}.$$

For any $x^b \in \mathrm{supp}(f_{i,j}) \cap K[\mathbf{x}_2, \ldots, \mathbf{x}_r]$, we have $w_0(\pi(b)) = -v(x^b)$. Since I satisfies Equation (†), for any $x^b \in \mathrm{supp}(f_{i,j}) \cap K[\mathbf{x}_2, \ldots, \mathbf{x}_r]$ we have $w_0(\pi(b)) \leq -1$.

Moreover, for all $2 \le i \le r-1, 1 \le j \le a_i$, we claim that there exists some $x^b \in \operatorname{supp}(f_{i,j}) \cap K[\mathbf{x}_i]$. Since I satisfies Equation (†), by Proposition 2.23 we have $\overline{\frac{I_2+\cdots+I_r+(\mathbf{x}_1)}{(\mathbf{x}_1)}} = \frac{\mathfrak{D}+(\mathbf{x}_1)}{(\mathbf{x}_1)}$. It follows that $\overline{\frac{I_2+(\mathbf{x}_1)}{(\mathbf{x}_1)}} = (\mathbf{x}_2)^{d_2}$. In particular, we have that $f_{2,j} \in K[\mathbf{x}_2] + (\mathbf{x}_1)$. As $f_{2,1}, \dots, f_{2,a_2}$ is a complete intersection mod $(\mathbf{x}_1)$, we conclude that $\operatorname{supp}(f_{2,j}) \cap K[\mathbf{x}_2] \ne 0$. For $3 \le i \le r-1$, we apply the same argument to the image of $f_{i,j}$ mod $(\mathbf{x}_1, \dots, \mathbf{x}_{i-1})$. Consequently, for all $2 \le i \le r-1, 1 \le j \le a_i$ we have

$$\max_{x^b \in \operatorname{supp}(f_{i,j}) \cap K[\mathbf{x}_2,\dots,\mathbf{x}_r]} w_0(\pi(b)) \ge \max_{x^b \in \operatorname{supp}(f_{i,j}) \cap K[\mathbf{x}_i]} w_0(\pi(b)) = -1. \tag{18}$$

For all $2 \le i \le r, (u_1, \dots, u_r) \in S_0^i$ we have

$$-1 = \left(-\frac{1}{d_1} - \frac{1 - u_1/d_1 - \cdots - u_r/d_r}{u_1}\right) u_1 - \frac{u_2}{d_2} - \cdots - \frac{u_r}{d_r} \ge w_0(u_1, \dots, u_r). \tag{19}$$

We define a monomial partial order $>_0$ by

$$x^b >_0 x^{b'} \iff w_0(\pi(b)) >_0 w_0(\pi(b')). \tag{20}$$

For $2 \le i \le r-1, 1 \le j \le a_i$, we set $g_{i,j,0} = \operatorname{in}_{>_0}(f_{i,j})$. We define $J_0 = (\mathbf{x}_1)^{d_1} + (g_{2,1,0}, \dots, g_{r-1,a_{r-1},0}) + \mathfrak{m}^{d_r}$. By Equations (18) and (19), for $2 \le i \le r-1$ we have

$$\operatorname{supp}(\operatorname{in}_{>_0}(f_{i,j})) = \{x^b \in \operatorname{supp}(f_{i,j}) : w_0(b) = -1\}. \tag{21}$$

We constructed J_0 to prune away all monomials $x^b \in \operatorname{supp}(f_{i,j})$ such that $v(x^b) > 1$ while ensuring $c(J_0) \le c(I)$ and $J_0 \not\subseteq \mathfrak{D}$. We now construct a sequence of ideals $J_1, \dots, J_s$ for some $s \in \mathbb{Z}^+$, each ideal J_{i+1} a modification of J_i. Suppose we have defined

$$J_k = (\mathbf{x}_1)^{d_1} + (g_{2,1,k}, \dots, g_{r,a_r,k}) + \mathfrak{m}^{d_r}.$$

For $k \ge 0$, define

$$T_{k+1}^{i,j} := \{\pi(b) : x^b \in \operatorname{supp}(g_{i,j,k}) \text{ and } \pi_1(b) \ne 0\}, \quad T_{k+1}^i = \bigcup_{j=1}^{a_i} T_k^{i,j}, \quad S_0 = \bigcup_{i=2}^{r-1} T_{k+1}^i. \tag{22}$$

As we proceed with our iterative construction, we ensure that the following properties are maintained:

(∗)
- (i) For all k, we assume that $\operatorname{supp}(f_{i,j}) \cap K[\mathbf{x}_i] \subseteq \operatorname{supp}(g_{i,j,k}) \subseteq \operatorname{supp}(f_{i,j})$. For $k \ge 1$, we further assume that $\operatorname{supp}(g_{i,j,k}) \cap K[\mathbf{x}_2, \dots, \mathbf{x}_r] = \operatorname{supp}(f_{i,j}) \cap K[\mathbf{x}_i]$.
- (ii) For all k, we assume that $T_{k+1} \ne \varnothing$.
- (iii) For all k, we assume that $\sum_{i=1}^r \frac{u_i}{d_i} < 1$ for all $(u_1, \dots, u_r) \in T_{k+1}$.

We verify the conditions $*$ for $k = 0$:

- (i) Follows from Equations (18), (19) and (21).
- (ii) We constructed w_0 so that Equation (19) is sharp for some $u \in T_0$.
- (iii) For all $(u_1, \dots, u_r) \in S_0 \setminus T_0$ we have

$$w_0(u_1, \dots, u_r) = -tu_1 - \sum_{i=1}^r \frac{u_i}{d_i} \le tu_1 - 1 < -1.$$

We begin the construction of J_1. Set

$$t_1 := \max_{2 \le i \le r-1} \max_{(u_1,\dots,u_r) \in T_1^i} \frac{d_r^2 u_2 + \cdots + d_r^r u_r - d_i d_r^i}{u_1}, \qquad w_1(u_1, \dots, u_r) = t_1 u_1 - d_r^2 u_2 - \cdots - d_r^r u_r. \tag{23}$$

Define $>_1$ analogously to Equation (20) and for $2 \leq i \leq r-1, 1 \leq j \leq a_i$, define $g_{i,j,1} = \text{in}_{>_1}(g_{i,j,0})$. We verify that J_1 satisfies the conditions $*$.

(i) For $2 \leq i \leq r-1, u \in T_1^i$, we have $w_1(u) \leq -d_i d_r^i$. Let $x^b \in \text{supp}(g_{i,j,0}) \cap K[\mathbf{x}_2, \dots, \mathbf{x}_r]$. If $x^b \in K[\mathbf{x}_i]$, then $w_1(\pi(b)) = -d_i d_r^i$. If $x^b \notin K[\mathbf{x}_i]$, then since $v(x^b) = 1$, there exists some $i < l \leq r, 1 \leq j \leq a_l$ such that $x_{l,j} \mid x^b$, hence

$$w_1(x^b) \leq w(x_{l,j}) \leq -d_r^{i+1} < -d_i d_r^i. \tag{24}$$

It follows that

$$\text{supp}(g_{i,j,1}) = \{x^b \in \text{supp}(g_{i,j,0} : w_1(\pi(b)) = -d_i d_r^i)\}. \tag{25}$$

(ii) Let $2 \leq i \leq r-1, u \in T_1^{i,j}$ such that u realizes the maximum in Equation (23). Then $w_1(u) = -d_i d_r^i$, so the monomial summand $x^b \in \text{supp}(g_{i,j,0})$ such that $\pi(b) = 0$ ties for the leading term, hence $T_2^{i,j} \neq 0$.

(iii) This follows from the fact that J_1 satisfies (i) and J_0 satisfies (iii).

Suppose now $k \geq 2$ and we have defined $J_0, \dots, J_{k-1}$. Let $\Lambda_k = \{2 \leq i \leq r-1 : T_2^i \neq \varnothing\}$. By assumption, $|\Lambda_k| \neq \varnothing$. If $|\Lambda_k| = 1$, then J_{k-1} will be our final ideal in the sequence $J_0, \dots, J_s$. Otherwise, we will construct an ideal J_k such that $|\Lambda_{k+1}| \subsetneq |\Lambda_k|$. Let $\lambda_k = \min \Lambda_k$. Let

$$t_k := \max_{i \in \Lambda_k \setminus \{\lambda_k\}, (u_1, \dots, u_r) \in T_k^i} \frac{u_{\lambda_k}}{u_1}, \quad w_k(u_1, \dots, u_r) := -t_k u_1 + u_{\lambda_k}. \tag{26}$$

We then let $>_k$ analogously to Equation (20), we set $g_{i,j,k} = \text{in}_{>_k}(g_{i,j,k-1})$ and define

$$J_k = (\mathbf{x}_1)^{d_r} + (g_{2,1,k}, \dots, g_{r-1,a_{r-1},k}) + \mathfrak{m}^{d_r}.$$

We verify that J_k satisfies properties (i)-(iii):

(i) For $i \in \Lambda_k \setminus \{\lambda_k\}, (u_1, \dots, u_r) \in T_k^i$, we have $w_k(u_1, \dots, u_r) \leq 0$. For $x^b \in \text{supp}(g_{i,j,k-1}) \cap K[\mathbf{x}_i]$, we have $w_k(\pi(b)) = 0$. As $\text{supp}(g_{i,j,k-1}) \cap K[\mathbf{x}_i] \neq \varnothing$, we have

$$\text{supp}(g_{i,j,k}) = \{x^b \in \text{supp}(g_{i,j,k-1}) : w_k(\pi(b)) = 0\}.$$

The result then follows from the fact that J_{k-1} satisfies (i).

(ii) As in the case $k = 1$, this follows from our choice of t_k.

(iii) This follows from condition (i) for J_k and condition (iii) for J_0.

We note one final comparison between J_{k-1}, J_k. For all $(u_1, \dots, u_r) \in T_k^{\lambda_k}$, we have $w_k(u_1, \dots, u_r) \leq u_{\lambda_k} < d_{\lambda_k} = w_k(C_k)$, so the equality is never achieved for any $(u_1, \dots, u_r) \in T_k^{\lambda_k}$. It follows that $\Lambda_{k+1} \subseteq \Lambda_k \setminus \{\lambda_k\}$, hence this sequence of ideals $J_1, \dots, J_s$ eventually terminates. Let $J = J_s$ denote the final ideal in this sequence, and write $h_{i,j} = g_{i,j,s}$. We now verify that J satisfies the five properties in the lemma statement.

(1) Follows from the fact that $\text{supp}(g_{i,j,s}) \subseteq \text{supp}(f_{i,j})$.
(2) Here, m is the unique element of Λ_s. This follows from condition (i) on J_s.
(3) Follows from the facts that $\text{supp}(g_{i,j,s}) \cap K[\mathbf{x}_2, \dots, x_r] = \text{supp}(f_{i,j}) \cap K[\mathbf{x}_i]$ and $T_{s+1}^i = \varnothing$.
(4) Follows from the fact that $\text{supp}(g_{i,j,s}) \cap K[\mathbf{x}_2, \dots, x_r] = \text{supp}(f_{i,j}) \cap K[\mathbf{x}_i]$.
(5) As $J_k \subseteq \text{in}_{>_k}(J_{k-1})$ for $k \geq 1$, we have by Proposition 2.7

$$c(I) = c(\overline{I}) \geq c(\text{in}_{>_0}(\overline{I})) \geq c(J_0) \geq c(\text{in}_>(J_0)) \geq c(J_1) \geq \cdots \geq c(J_s) = c(J).$$

□

Lemma 4.6. *Assume the setting of Definition 4.1. Let $J \subseteq R$ be an ideal satisfying (1)-(4) of Lemma 4.5. Then $c(J) > DP(J)$.*

Proof. Set $J_m = (h_{m,1}, \dots, h_{m,a_m})$. Let $J' = (\mathbf{x}_1)^{d_1} + \dots + J_m + (\mathbf{x}_{m-1})^{d_{m-1}} + +(\mathbf{x}_{m+1})^{d_{m+1}} + \dots + (\mathbf{x}_r)^{d_r}$. By (2) and (4), we have $\overline{J} = \overline{J'}$, so it suffices to show $c(J') > DP(J')$. We first prove this result in characteristic $p > 0$.

By assumption, $J_m \not\subseteq \mathfrak{D}$. Let σ denote the maximum value of $v(x^b)$ over all $x^b \in \text{supp}(I) \backslash (\mathbf{x}_m)^{d_m}$, which satisfies $\sigma < 1$ by condition (2). By Theorem [1], Theorem 3.17 we have $c(J_m) > \frac{a_m}{d_m}$. Let f be a generator of $(J_m)^{\nu_{J_m}(p^e)}$ such that $f \notin \mathfrak{m}^{[p^e]}$. Write

$$f = \sum_{b \in \text{supp}(f)} \alpha_b x^b, \quad f' := \sum_{b \in \text{supp}(f) : x^b | (\mathbf{x}_1 \cdots \mathbf{x}_r)^{p^e - 1}} \alpha_b x^b.$$

As $f \equiv f' \mod \mathfrak{m}^{[p^e]}$, we have $f' J_m \subseteq \mathfrak{m}^{[p^e]}$. Let $\succ$ denote the negative lexicographic order after permuting the variables into the order $\mathbf{x}_m, \mathbf{x}_1, \dots, \mathbf{x}_{m-1}, x_{m+1}, \dots, \mathbf{x}_r$. By Brianǫn-Skoda on the ideal $\frac{J_m + (\mathbf{x}_1)}{(\mathbf{x}_1)}$, we have $(\mathbf{x}_m)^{a_m d_m} \subseteq \text{in}_\succ(J_m)$. Let $\text{in}_\succ(f') = \alpha_b x^b$. Since $f' J_m \subseteq \mathfrak{m}^{[p^e]}$, taking initial terms we also have $\text{in}_\succ(f')(\mathbf{x}_m)^{a_m d_m} \subseteq \mathfrak{m}^{[p^e]}$. By Lemma 3.15, we have

$$\text{ord}_{\mathbf{x}_m}(x^b) \geq a_m(p^e - 1) - a_m d_m \tag{27}$$

Consequently, we have

$$v(f') = v(x^b) \leq \left\lfloor \frac{a_m(p^e - 1) - a_m d_m}{d_m} \right\rfloor + \sigma \left(\nu_{J_m}(p^e) - \left\lfloor \frac{a_m(p^e - 1) - a_m d_m}{d_m} \right\rfloor \right). \tag{28}$$

As in Lemma 3.15, let $\mathfrak{a}_\lambda$ denote the ideal $\{f \in R : v(f) \geq \lambda\}$. Let t_e denote the right-hand side of Equation (28) and set $u_e := (p^e - 1)(\frac{a_1}{d_1} + \dots + \frac{a_r}{d_r})$. It follows from Lemma 3.15 that

$$f' \notin \mathfrak{m}^{[p^e]} + \mathfrak{a}_{t_e + 1} = (\mathfrak{m}^{[p^e]} : \mathfrak{a}_{u_e - t_e - 1}).$$

Let $x^{b'} \in \mathfrak{a}_{u_e - t_e - 1}$ such that $x^{b + b'} \notin \mathfrak{m}^{[p^e]}$. Let $x^{b''}$ denote the largest factor of $x^{b'}$ such that $x^{b''} \notin (\mathbf{x}_m)$. By Equation (27), we must have $\text{ord}_{(\mathbf{x}_m)}(x^{b' - b''}) \leq a_m d_m$, hence $v(x^{b''}) \geq v(x^{b''}) - a_m \geq \lfloor u_e - t_e \rfloor - 1 - a_m$. As $\mathfrak{D} = \overline{(\mathbf{x}_1)^{d_1} + \dots + (\mathbf{x}_r)^{d_r}}$, by Briançon-Skoda we have

$$x^{b''} \in \mathfrak{D}^{\lfloor u_e - t_e \rfloor - 1 - a_m - n} \subseteq ((\mathbf{x}_1)^{d_1} + \dots + (\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - 1 - a_m - n}.$$

Since $x^{b''} \notin (\mathbf{x}_m)$, we in fact have

$$x^{b''} \in ((\mathbf{x}_1)^{d_1} + \dots + (\mathbf{x}_{m-1})^{d_{m-1}} + (\mathbf{x}_{m+1})^{d_{m+1}} (\mathbf{x}_r)^{d_r})^{\lfloor u_e - t_e \rfloor - 1 - a_m - n} \subseteq (J')^{\lfloor u_e - t_e \rfloor - 1 - a_m - n}.$$

It follows that $\nu_{J'}(p^e) \geq \nu_{J_m}(p^e) + \lfloor u_e - t_e \rfloor - 1 + a_m - n$. Dividing by p^e and letting $e \to \infty$, we obtain

$$\begin{aligned} c(J') &\geq c(J_m) + \lim_{e \to \infty} \frac{u_e}{p^e} - \lim_{e \to \infty} \frac{t_e}{p^e} \\ &= c(J_m) + \left(\frac{a_1}{d_1} + \dots + \frac{a_r}{d_r} \right) - \left(\frac{a_m}{d_m}(1 - \sigma) + \sigma c(J_m) \right) \\ &= (1 - \sigma) \left(c(J_m) - \frac{a_m}{d_m} \right) + \left(\frac{a_1}{d_1} + \dots + \frac{a_r}{d_r} \right) \end{aligned}$$

Since $\sigma < 1$ and $c(J_m) > \frac{a_m}{d_m}$, it follows that the above quantity exceeds $DP(J')$.

In characteristic zero, one notes that for any ideal J satisfying conditions (1)-(4), the reduction of the pair (R, J) to characteristic $p \gg 0$ satisfies conditions (1)-(4). Moreover, the quantity σ is constant for $p \gg 0$. Assuming the reduction notation of Definition 3.13, we have

$$c(J) = \lim_{\substack{\mu \in \text{Spec } A \\ |A/\mu| \to \infty}} c(J_\mu) \geq (1 - \sigma) \lim_{\substack{\mu \in \text{Spec } A \\ |A/\mu| \to \infty}} c(J_{m,\mu}) + DP(J) = (1 - \sigma) c(J_m) + DP(J) > DP(J).$$

□

Lemmas 4.4 to 4.6 combine to give us a proof of Theorem 4.14 in the case of a complete intersection.

Proposition 4.7. *Assume the setup of Definition 4.1 and suppose $c(I) = DP(I)$. Then there exists $\varphi \in \mathrm{GL}_n(K)$ depending only on $I_1, \dots, I_{r-1}$ such that $\overline{\varphi^* I} = \mathfrak{D}$.*

Proof. Using Lemma 4.4, we produce $\varphi \in \mathrm{GL}_n(K)$ such that $\varphi^* I$ satisfies Equation (†). By Lemmas 4.5 and 4.6, we have $\overline{\varphi^* I} = \mathfrak{D}$. □

4.2. **Generalizations.**

Lemma 4.8. *Let $R = K[x_1, \dots, x_n]$ and let $\mathfrak{m} = (x_1, \dots, x_n)$. Let $I \subseteq R$ be a homogeneous ideal and $J \subseteq \mathfrak{m}$ any ideal. Then we have*

$$\bigcap_{m>0} \overline{I + J^m} = \overline{I}.$$

Proof. By [21, Corollary 6.8.5], we have

$$\overline{IR_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \subseteq \bigcap_{m>0} \overline{IR_\mathfrak{m} + \mathfrak{m}^m R_\mathfrak{m}} = \overline{IR_\mathfrak{m}}.$$

As $\overline{I} = \overline{I} R_\mathfrak{m} = \overline{IR_\mathfrak{m}} \cap R$, we have the following, from which the claim follows.

$$\overline{I} \subseteq \bigcap_{m>0} \left(\overline{IR_\mathfrak{m} + J^m R_\mathfrak{m}} \cap R \right) \subseteq \overline{IR_\mathfrak{m}} \cap R = \overline{I}.$$

□

The fact that φ depends only on $I_1, \dots, I_{r-1}$ allows us to prove a version of Theorem 4.14 for complete intersections of smaller codimension.

Proposition 4.9. *Let K be an algebraically-closed field and set $R = K[x_1, \dots, x_n]$. Set $\mathfrak{m} = (x_1, \dots, x_n)$. Suppose $r < n$ and $I = (f_1, \dots, f_r)$ is a complete interesection, where f_i is homogeneous of degree d_i and $d_1 \leq \dots \leq d_r$. Then $c(I) \geq \frac{1}{d_1} + \dots + \frac{1}{d_r}$ with equality if and only if there exists $\varphi \in \mathrm{GL}_n(K)$ such that*

$$\varphi^* \overline{I} = \overline{(x_1^{d_1}, \dots, x_r^{d_r})}.$$

Proof. Let $L = (\ell_{r+1}, \dots, \ell_n)$ be an ideal generated by $n - r$ linear forms. Then $\frac{I+L}{L}$ is a complete intersection of type $(d_1, \dots, d_r)$ which is primary to $\frac{\mathfrak{m}}{L}$, so we have

$$c(R, I) \geq c\left(\frac{R}{L}, \frac{I+L}{L}\right) \geq DP\left(\frac{I+L}{L}\right) = \frac{1}{d_1} + \dots + \frac{1}{d_r}.$$

Suppose now that $c(I) = \frac{1}{d_1} + \dots + \frac{1}{d_r}$. Let $\ell_{r+1}, \dots, \ell_n$ be general linear forms such that $(f_1, \dots, f_r, \ell_{r+1}, \dots, \ell_r)$ is a complete intersection.

For each $e > d_r$, let $J_e := (\ell_{r+1}^e, \dots, \ell_n^e)$. Then $I + J_e$ is a complete intersection of $(d_1, \dots, d_r, e, \dots, e)$. It follows that

$$\frac{1}{d_1} + \dots + \frac{1}{d_r} + \frac{n-r}{e} = DP(I + J_e) \leq c(I + J_e) \leq c(I) + c(J_e) = \frac{1}{d_1} + \dots + \frac{1}{d_r} + \frac{n-r}{e}.$$

By Proposition 4.7, there exists $\varphi \in \mathrm{GL}_n(K)$ such that for all $e > d_r$, we have $\varphi^* \overline{(I + J_e)} = \overline{(x_1^{d_1}, \dots, x_r^{d_r}, x_{r+1}^e, \dots, x_n^e)}$. By Lemma 4.8, we conclude

$$\varphi^* \overline{I} = \bigcap_{e>d_r} \varphi^* \overline{I + J_e} = \bigcap_{e>d_r} = \overline{(x_1^{d_1}, \dots, x_r^{d_r})}.$$

□

Lemma 4.10. *Let K be an uncountably infinite field. Let $R = K[x_1, \dots, x_n]$ and set $\mathfrak{m} = (x_1, \dots, x_n)$. Suppose $I \subseteq R$ is a homogeneous ideal. As in Lemma 3.1, for $1 \le j \le n$, let $\pi_j : R \to R/(x_{j+1}, \dots, x_n) \cong K[x_1, \dots, x_j]$ denote the projection map and $\iota_j : K[x_1, \dots, x_j] \to K[x_1, \dots, x_n]$ the usual embedding. Let $>$ denote the reverse lexicographic order.*

Let $\varphi \in \mathrm{GL}_n(K)$ be very general: for now, we impose the condition that for all $m > 0$, we have $\mathrm{in}_>(\varphi^* I^m) = \mathrm{gin}_>(I^m)$; we will impose countably many additional conditions in Lemma 4.13. For $1 \le j \le n, m > 0$, set $\mathfrak{a}_{j,m} := \mathrm{in}_>(\pi_j(\varphi^* I^m))$. For $j > 0, 1 \le i \le j$, let $b_{i,j}$ denote the ith unit vector of $\mathbb{R}^j$. Set $p_j(i) := \inf\{t : tb_{i,j} \in \Gamma(\mathfrak{a}_{j,\bullet})\}$. Then for all j, we have $p_j(j) = p_n(j)$.

Proof. By Lemma 3.1, we have $\iota_j(\mathfrak{a}_{j,\bullet}) \subseteq \mathfrak{a}_{n,\bullet}$ for all $1 \le j \le n$, so we have $p_j(j) \le p_n(j)$. For the reverse direction, set $t = p_n(j)$. Since $tb_{j,n} \in \overline{\bigcup_{m>0} \frac{1}{2^m}\Gamma(\mathfrak{a}_{n,2^m})}$, there exists a sequence $\{a_m = (a_{m,1}, \dots, a_{m,n})\}_{m>0}$ such that $a_m \in \Gamma(\mathfrak{a}_{n,2^m})$ for all m and $\lim_{m\to\infty} 2^{-m} a_m = tb_{j,n}$. For any choice of $\{(a_{m,1}, \dots, a_{m,n})\}_{m>0}$, we also have $(\lceil a_{m,1}\rceil, \dots, \lceil a_{m,n}\rceil) \in \Gamma(\mathfrak{a}_{n,2^m})$ and $\lim_{m\to\infty} \frac{(\lceil a_{m,1}\rceil, \dots, \lceil a_{m,n}\rceil)}{2^m} = tb_{j,n}$. We may therefore assume without loss of generality that $a_m \in (\mathbb{Z}^+)^n$ for all $m > 0, 1 \le i \le n$, hence for all $m > 0$, we have $x^{a_m} \in \overline{\mathfrak{a}_{n,2^m}}$.

By [12, Theorem 2.1], $\overline{\mathfrak{a}_{n,2^m}}$ is Borel-fixed, so we have $x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j} + \dots + a_{m,n}} \in \overline{\mathfrak{a}_{n,2^m}}$. By Proposition 2.27(iii) and (vii), we have

$$x_1^{a_{m,1}} \cdots x_{j-1}^{a_{m,j-1}} x_j^{a_{m,j} + \dots + a_{m,n}} \in (\iota_j \circ \pi_j)(\overline{\mathfrak{a}_{n,2^m}}) \subseteq \iota_j\left(\overline{\pi_j(\mathfrak{a}_{n,2^m})}\right) = \overline{\mathfrak{a}_{j,m}}. \tag{29}$$

It follows that

$$tb_{j,j} = \lim_{m\to\infty} \frac{(a_{m,1}, \dots, a_{j-1}, a_j + \dots + a_n)}{m} \in \Gamma(\mathfrak{a}_{j,\bullet}),$$

which proves $p_n(j) \le t = p_n(j)$. □

Lemma 4.11. *Assume the setup of Lemma 4.10. There exists a sequence $\{a'_m\}_{m>0}$ such that for all $m > 0$, we have $x^{a'_m} \in \mathfrak{a}_{j,2^m}$ and $\lim_{m\to\infty} 2^{-m} a'_m = p_j(j) b_{j,j}$.*

Proof. Consequently, there exists a sequence $\{a_m = (a_{m,1}, \dots, a_{m,j})\}_{m>0}$ such that for all $m > 0$ we have $\lim_{m\to\infty} 2^{-m} a_m = tb_{j,j}$ and $a_m \in \Gamma(\mathfrak{a}_{2^m})$.

First, note that $\Gamma(\mathfrak{a}_{2^m}) = \mathrm{conv}(\log(x^u) : x^u \in \mathfrak{a}_{2^m})$. If we triangulate $\Gamma(\mathfrak{a}_{2^m})$, we may choose $\{u_{m,i} = (u_{m,i,1}, \dots, u_{m,i,j})\}_{i=0}^{j}$ such that $x^{u_{m,i}} \in \mathfrak{a}_{2^m}$ and $a_m \in \mathrm{conv}(u_{m,0}, \dots, u_{m,j+1})$. Reorder the $u_{m,i}$ so that $u_{m,0,j} \le \dots \le u_{m,j,j}$. Since $a_{m,j}$ is the average of the $u_{m,i,j}$, we have $u_{m,0,j} \le a_{m,j}$. For $i < j$, we similarly have $u_{m,0,i} \le (j+1)a_{0,i}$.

For all $m > 0$, set $a'_m = u_{m,0}$. Then we have $\lim_{m\to\infty} a'_{m,i} = 0$ for all $i < j$, and

$$t \le \liminf_{m\to\infty} 2^{-m} a'_{m,j} \le \lim_{m\to\infty} 2^{-m} a_{m,j} = t.$$

It follows that $\lim_{m\to\infty} 2^{-m} a'_m = tb_{j,j}$ and for all $m > 0, x^{a'_m} \in \mathfrak{a}_{2^m}$. □

Lemma 4.12. *Let K be an algebraically-closed field and $R = K[x_1, \dots, x_j]$. Let $\mathfrak{q}$ be a homogeneous prime ideal of codimension $j-1$ with $x_j \notin \mathfrak{q}$. If $>$ denotes the reverse lexicographic order, then for all $m > 0$ we have $\mathrm{in}_>(\mathfrak{q}^m) = (x_1, \dots, x_{j-1})^m$.*

Proof. Since K is algebraically-closed, there exist linear forms $\ell_1, \dots, \ell_{j-1} \in R_1$ such that $q = (\ell_1, \dots, \ell_{j-1})$. It follows that $[\mathrm{in}_>(\mathfrak{q})]_1$ is generated by $\mathrm{in}_>(\ell_1 \wedge \dots \ell_{j-1})$, which is equal to $x_1 \wedge \dots \wedge x_{j-1}$ by the fact that $x_j \notin \mathrm{span}(\ell_1, \dots, \ell_{j-1})$. Consequently, we have $(x_1, \dots, x_j) \subseteq \mathrm{in}_>(\mathfrak{q})$. By [8, Theorem 15.17], $\mathrm{in}_>(\mathfrak{q})$ and $\mathfrak{q}$ have the same Hilbert series, so we in fact have $(x_1, \dots, x_j) = \mathrm{in}_>(\mathfrak{q})$.

For $m > 1$, a similar analysis applies. We have the standard containment $(x_1, \dots, x_{j-1})^m = \mathrm{in}_>(\mathfrak{q})^m \subseteq \mathrm{in}_>(\mathfrak{q}^m)$. As $(x_1, \dots, x_{j-1})^m$ has the same Hilbert series as $\mathfrak{q}$, the result follows. □

Lemma 4.13. *Assume the setup of Lemmas 4.10 and 4.11. Further assume that $K = \overline{K}$. Write $I = I_1 + \dots + I_r$, where each I_i is generated by d_i-forms and $d_1 < \dots < d_r$. For $1 \le j \le r$, set*

$h_i := \operatorname{codim}(I_1 + \cdots + I_i) - \operatorname{codim}(I_1 + \cdots + I_{i-1})$. *For* $1 \le i \le n$, *we also define* $q_i := j$, *where* $1 \le j \le r$ *such that* $h_1 + \cdots + h_{j-1} < i \le h_1 + \cdots + h_j$. *Then for all* $1 \le j \le n$, *we have* $p_n(j) = d_{q_j}$.

Proof. Before we begin the proof, we first state the additional generality conditions on φ. For all $m > 0$, assume that $\operatorname{in}_>(\pi_j(\varphi^* I^m)) = \operatorname{gin}_>(\pi_j(\varphi^* I^m)) = \pi_j(\operatorname{in}_>(\varphi^* I^m))$; this is possible by repeated application of [3, Theorem 1.13.]. Since $\operatorname{codim} \pi_j(\varphi^*(I_1 + \cdots + I_{q_j - 1})) < j$, we may also choose φ such that $x_j \notin \sqrt{\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))}$. Finally, since $\operatorname{codim} \pi_j(I_1 + \cdots + I_{q_j}) \ge j$, we may choose φ such that $\pi_j(\varphi^*(I_1 + \cdots + I_{q_j}))$. Each of these conditions is satisfied by a general choice of φ, so they may be realized simultaneously.

Set $J = \pi_j(\varphi^* I)$. By construction of φ, in the language of Lemma 4.10 we have $\mathfrak{a}_{j,m} = \operatorname{in}_>(J^m)$. By construction of φ, we have $x_j \notin \sqrt{\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))}$, so we may choose a minimal prime $\mathfrak{p}$ over $\pi_j(\varphi^*(I_1 + \cdots + I_{q_j-1}))$ such that $x_j \notin \mathfrak{p}$. As $\operatorname{codim} \mathfrak{p} \le j-1$, we may choose a homogeneous prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$ such that $\operatorname{codim} \mathfrak{q} = j-1$ and $x_j \notin \mathfrak{q}$. By Lemma 4.12, we have $\operatorname{in}_>(\mathfrak{q}^m) = (x_1, \ldots, x_{j-1})^m$ for all $m > 0$.

By Lemma 4.11, choose a sequence $\{a_m\}_{m>0}$ such that $x^{a_m} \in \mathfrak{a}_{2^m}$ for all $m > 0$ and $\lim_{m\to\infty} 2^{-m} a_m = p_j(j) b_{j,j}$. Let $e_m := a_{m,1} + \cdots + a_{m,j}$. For all $m > 0$, we have

$$[J^{2^m}]_{e_m} = \left[\sum_{\substack{\gamma_1 + \cdots + \gamma_r = 2^m \\ \gamma_1 d_1 + \cdots + \gamma_r \le e_m}} \pi_j(\varphi^* I_1)^{\gamma_1} \cdots I_r^{\gamma_r} \right]_{e_m}$$

For $1 \le i \le q_j - 1$, we have $I_i \subseteq \mathfrak{q}$. For $q_j \le i \le r$, we have $I_i \subseteq (x_1, \ldots, x_j) = \mathfrak{m}$. It follows that

$$[J^{2^m}]_{e_m} \subseteq \left[\sum_{\substack{\alpha + \beta = 2^m \\ \beta d_{q_j} \le e_m}} \mathfrak{q}^\alpha \mathfrak{m}^\beta \right]_{e_m} \subseteq \left[\mathfrak{q}^{2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor} \right]_{e_m}.$$

Taking initial ideals of both sides, we have $x^{a_m} \in (x_1, \ldots, x_j)^{2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor}$. Consequently, we have $e_m - a_{m,j} = a_{m,1} + \cdots + a_{m,j-1} \ge 2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor$. As $\lim_{m\to\infty} 2^{-m}(a_{m,1} + \cdots + a_{m,j-1}) = 0$, this yields

$$0 \le \liminf_{m\to\infty} 2^{-m} \left(2^m - \left\lfloor \frac{e_m}{d_{q_j}} \right\rfloor \right) = 1 - \frac{1}{d_{q_j}} \liminf_{m\to\infty} \frac{a_{m,j}}{d_{q,j}} = 1 - \frac{p_j(j)}{d_{q_j}}.$$

From the above equation, we have $p_j(j) \ge d_{q_j}$. For the reverse containment, we have by Lemma 3.9 that $\mathfrak{m}^{d_{q_j}} \subseteq \overline{J}$. It follows that $x_j^{m+j-1} \in J^m$ for all $m > 0$, hence $p_j(j) \le d_{q_j}$. □

We are now able to prove Theorem 4.14.

Theorem 4.14. *Let* K *be an algebraically-closed field. Let* $R = K[x_1, \ldots, x_n]$ *and let* $I \subseteq R$ *be a* $\mathfrak{m}$-*primary homogeneous ideal. If* $DP(I) = c(I)$, *then there exist integers* $d_1, \ldots, d_n$ *and* $\varphi \in \mathrm{GL}_n(K)$ *such that*

$$\varphi^* \overline{I} = \overline{\left(x_1^{d_1}, \ldots, x_n^{d_n} \right)}.y$$

Proof. By Corollary 3.11, we have

$$\overline{\mathbb{R} \setminus \Gamma_>(\mathfrak{a}_\bullet)} = \operatorname{conv}\left(\vec{0}, (e_1(I), 0, \ldots, 0), \left(0, \frac{e_2(I)}{e_1(I)}, 0, \ldots, 0\right), \ldots, \left(0, \ldots, 0, \frac{e_n(I)}{e_{n-1(I)}}\right) \right).$$

Assume the h_i, q_i notation from Lemma 4.13. If we let $L := \overline{K((t))}$, then L is uncountably infinite and algebraically closed. The generic initial ideal is stable under field extension, so applying Lemma 4.13 to $I \otimes_K L$, we have $\frac{e_j(I)}{e_{j-1}(I)} = d_{q_j}$ for all $1 \le j \le n$.

Let $J \subseteq I$ be an ideal generated by h_i general elements of I_i for each $1 \le i \le r$. Then J is a homogeneous $(d_{q_1}, \dots, d_{q_n})$-complete intersection, so by Lemma 3.10, we have $DP(J) = DP(I)$. It follows from Proposition 2.23 that $\overline{J} = \overline{I}$, so we have $c(J) = c(I) = DP(I) = DP(J)$. By Proposition 4.7, there exists $\varphi \in \mathrm{GL}_n(K)$ such that

$$\varphi^*\overline{I} = \varphi^*\overline{J} = \overline{\left(x_1^{d_{q_1}}, \dots, x_n^{d_{q_n}}\right)}.$$

□

5. Future Work

We can restate Theorem 4.14 as follows.

Proposition 5.1. *Let $R = K[x_1, \dots, x_n], \mathfrak{m} = (x_1, \dots, x_n)$, and let I be an $\mathfrak{m}$-primary ideal. Then $\overline{I} = (x_1^{d_1}, \dots, x_n^{d_n})$ in suitable coordinates if and only if $DP(I) = \mathrm{lct}(I)$ and $\overline{I}$ is a homogeneous ideal.*

We conjecture that the second condition is unnecessary.

Conjecture 5.2. *Assume the setup of Proposition 5.1. If $DP(I) = \mathrm{lct}(I)$, then $\overline{I}$ is a homogeneous ideal.*

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