Classification of Minimal Singularity Thresholds

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Notation

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- k an algebraically-closed field
- $R = k[x_1, \ldots, x_n]$
- $\mathfrak{m} = (x_1, \ldots, x_n)$

Multiplicity and the Log Canonical Threshold

[DFEM04, Theorems 0.1, 1.4]

Let char $k = 0$ and let I be an $\mathfrak{m}$-primary ideal. Then

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This bound is sharp, but it can be refined!

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- Standard definition in terms of the colengths of $I^r\mathfrak{m}^s$ for $r, s \in \mathbb{Z}^+$.

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If $I \subseteq J$, then $\mathrm{DP}(I) \leq \mathrm{DP}(J)$ with equality iff $\overline{I} = \overline{J}$.

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[DP14, Theorem 1.2], algebraic restatement

- If $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is $\mathfrak{m}$-primary, then $ne(I)^{-1/n} \leq \mathrm{DP}(I) \leq \mathrm{lct}(I)$.

Progress Towards a Classification

- Demailly-Pham 2014: if $J = (x_1^{d_1}, \ldots, x_n^{d_1})$, then $\mathrm{DP}(J) = \mathrm{lct}(J)$.

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Theorem 1 (B. 2025):

Let char $k = 0$ and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. If I is homogeneous and $\mathrm{lct}(I) = \mathrm{DP}(I)$, then

$$I = (x_1^{e_1(I)/e_0(I)}, \ldots, x_n^{e_n(I)/e_{n-1}(I)})$$

up to change of variables and integral closure.

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Proposition (B. 2025)

Let char $k = p > 0$ and $I \subseteq R$ an $\mathfrak{m}$-primary ideal. Then $\mathrm{DP}(I) \leq \mathrm{fpt}(I)$.

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Future Work

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References

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